An Axiomatically Consistent Single-Factor Term Structure Model with Admissible Humped Volatility

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Abstract

We develop an axiomatic framework for the dynamics of the instantaneous rolling forward rate (IRFR) and the associated term structure of interest rates. Rather than postulating a stochastic process directly, admissible dynamics are derived from structural principles: rolling-maturity invariance, equilibrium transport of the first two moments, affine closure of the Kolmogorov Forward Equation, and positivity at zero maturity. These conditions are consistent with an IRFR process governed by a Pearson-type diffusion with affine drift and quadratic volatility.

The framework admits a maturity-indexed equilibrium regime in which the forward curve converges exponentially to a long-maturity level and the equilibrium density is a shifted inverse-gamma law. Although equilibrium transport is imposed axiomatically only on the first two moments, the explicit KFE solution provides a full maturity-indexed equilibrium density, so that all finite moments admitted by that density inherit the same equilibrium structure. Crucially, the exponential scale governing maturity decay need not coincide with the temporal drift–diffusion scale. When these scales differ, the model generates two deterministic maturity loadings. These loadings can produce monotone or humped yield curves, with humped shapes arising outside the stationary equilibrium benchmark, and can also generate humped volatility term structures for both forward and conventional rates.

The interaction of the two scales localizes variance away from the short end without introducing additional stochastic factors. The stationary equilibrium volatility profile remains monotone; the humped volatility result is therefore an admissibility result for the non-stationary rolling-maturity branch rather than a claim about stationary one-factor volatility. The model further implies a constant market signal-to-noise ratio and a forward-curve transition structure analogous to affine covariance kernels. Together, these results provide a structural explanation for humped volatility and yield phenomena within a parsimonious rolling-rate framework.

Equilibrium terminology. Throughout, we work with the maturity-indexed equilibrium regime. All distributional results are expressed as maturity-indexed equilibrium densities parameterized by $\tau = T - t$, rather than as limits in calendar time.

1 Introduction

A central modelling choice in any term-structure framework is the selection of a rate-like quantity whose dynamics are both analytically tractable and economically meaningful. This paper adopts as its primitive the *instantaneous rolling forward rate* (IRFR).

Consider a forward-starting infinitesimal borrowing–lending contract initiated at time t_0 with zero initial value. At maturity $T - t_0$, the buyer pays an instantaneous interest rate $x_{t_0,T}$ on a notional principal. This value defines the IRFR. Because the contract is continuously marked to market, its value process is, by construction,

$$V(t) = x_{t,T},$$

and over an infinitesimal interval Δt,

$$\Delta V = x_{t+\Delta t,T} - x_{t,T} = \Delta x_{t,T}.$$

As time advances from t to $t + \delta t$ while the maturity date T remains fixed, the remaining maturity decreases from $T - t$ to $T - (t + \delta t)$. Consequently, the IRFR rolls through remaining maturity. This rolling structure is a defining feature of the IRFR and distinguishes it from a modelling convention in which all maturity variation is treated as an externally specified cross-sectional curve.

In continuous time, the IRFR satisfies the fundamental bond-pricing relations

$$\int_{t_0}^{T} x_{t_0,S}\, dS = -\ln P(t_0, T), \qquad \int_{t}^{T} x_{t,S}\, dS = -\ln P(t, T),$$

where $P(t, T)$ denotes the zero-coupon bond price and $T - t$ is the current time-to-maturity. Conventional continuously compounded rates are therefore obtained by maturity-averaging the IRFR curve. Appendix C uses this relation to transfer the general r-branch expectation and local covariance-rate formulas to conventional rates.

The objective of this paper is to characterise a structurally constrained class of stochastic processes governing the evolution of $x_{t,T}$. The starting axioms impose coherence across maturities and time, encode equilibrium moment transport, and enforce positivity at the short end. Affine closure and the drift–diffusion identification $a_1 = a^2$ then lead to a tractable IRFR specification. Importantly, the maturity-decay scale λ^2 need not be identified with a^2. We therefore retain the dimensionless ratio

$$r := \frac{\lambda^2}{a^2} > 0.$$

The resulting general r-framework has the following consequences:

- Under affine closure, the drift and diffusion of the IRFR are affine in the IRFR state, leading to a Pearson-type diffusion with quadratic volatility.

- The equilibrium forward curve has exponential-decay form

$$x_{e,T-t} = x_\infty \left(1 - \frac{1}{1+r} e^{-ra^2(T-t)}\right).$$

- The zero-maturity equilibrium forward rate is

$$x_{e,0} = \frac{r}{1+r} x_\infty,$$

  so the one-half result is the special case $r = 1$.

- The Kolmogorov Forward Equation admits a shifted inverse-gamma equilibrium density with shape parameter $r+2$ and scale parameter $rx_\infty e^{-ra^2(T-t)}$. The order-three density is the branch $r = 1$.

- The forward curve can be reconstructed from the observed short-end IRFR, x_∞, a, and r. For $r \neq 1$ this reconstruction has two exponential maturity loadings and can generate humped term structures in a one-factor model.

Humped volatility term structures are a well-documented empirical feature of interest-rate markets. Early empirical and option-pricing studies found that forward-rate and caplet volatilities often peak at intermediate maturities rather than declining monotonically [5, 2]. Ritchken and Chuang develop an HJM-based pricing framework with stationary humped forward-rate volatilities and report empirical support from caplet data [5]. Moraleda and Vorst construct a one-state-variable model allowing humped volatility structures for American interest-rate claims [4], while Mercurio and Moraleda develop analytically tractable humped-volatility specifications for interest-rate derivatives [2, 3]. More recent empirical HJM work finds that humped volatility specifications improve estimation and cap-pricing performance relative to strictly decreasing volatility structures [1]. The humped volatility result should be understood as an admissibility result for the non-stationary rolling-maturity branch, not as a claim that the stationary equilibrium volatility curve is humped.

2 Axioms

This section presents the structural axioms that constrain the dynamics of the instantaneous rolling forward rate. Each axiom has both an economic interpretation and a mathematical implication, and together they restrict the class of admissible diffusion processes governing $x_{t,T}$. The later baseline specification adds affine-closure and identification choices to obtain the explicit model studied in the remainder of the paper.

Axiom 1 — Stochastic Evolution

At every remaining maturity, the IRFR follows a continuous-state diffusion with density $\varphi(x,t,T)$ satisfying the Kolmogorov Forward Equation (KFE):

$$\frac{\partial \varphi}{\partial t} = -\frac{\partial}{\partial x}(\mu(x,t,T)\varphi) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(x,t,T)\varphi\right), \qquad T-t>0. \tag{2.0.1}$$

Axiom 2 — Rolling–Maturity Invariance

If the drift or diffusion depended directly on calendar time, then two IRFRs with the same state and remaining maturity could exhibit different instantaneous dynamics solely because they are implemented on different calendar dates. Unless such differences are offset elsewhere in the bond-price dynamics or in risk premia, this would introduce rolling implementation dependence.

To impose rolling consistency directly, the IRFR with remaining maturity $\tau = T - t$ is required to satisfy a KFE whose drift and diffusion depend only on (x, τ):

$$\frac{\partial \varphi}{\partial t} = -\frac{\partial}{\partial x}\big(\mu(x,\tau)\varphi\big) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\big(\sigma^2(x,\tau)\varphi\big), \qquad \tau > 0. \tag{2.0.2}$$

This enforces an invariance structure across calendar time: IRFRs with identical states and remaining maturities are assigned the same local stochastic behaviour within the model.

Axiom 3 — Equilibrium Moment Transport

There exists an equilibrium maturity curve $x_{e,T-t}$ such that, at equilibrium, the first two moments of the instantaneous rolling forward rate are transported along constant remaining maturity. In particular,

$$\frac{\partial}{\partial t}\,\mathbb{E}[x_{t,T}] = -\frac{\partial}{\partial T}\,\mathbb{E}[x_{t,T}]. \tag{2.0.3}$$

and

$$\frac{\partial}{\partial t}\,\mathbb{E}[x_{t,T}^2] = -\frac{\partial}{\partial T}\,\mathbb{E}[x_{t,T}^2]. \tag{2.0.4}$$

Equivalently, the variance satisfies

$$\frac{\partial}{\partial t}\,\mathrm{Var}(x_{t,T}) = -\frac{\partial}{\partial T}\,\mathrm{Var}(x_{t,T}). \tag{2.0.5}$$

The long-maturity boundary condition is

$$\lim_{T-t\to\infty} x_{e,T-t} = x_\infty.$$

This axiom imposes equilibrium transport of the mean and variance of the forward rate across maturity, rather than full stationarity of all moments.

Axiom 4 — Positivity at Zero Maturity

At zero time-to-maturity, the IRFR satisfies $x_{t,t} > 0$ almost surely. To ensure that the boundary at zero is inaccessible, the diffusion coefficient must vanish as the short-maturity IRFR approaches zero:

$$\sigma(x,t,t) \longrightarrow 0 \quad \text{as} \quad x_{t,t} \to 0.$$

3 Structural Framework

This section derives the structural implications of the axioms for the dynamics of the instantaneous rolling forward rate. Beginning from the Kolmogorov Forward Equation, we derive affine forms for the drift and diffusion of the IRFR and identify the associated equilibrium relation between the coefficients. The deterministic evolution of the expectation, the variance structure, and several key identities follow naturally.

3.0 KFE Representation and Notational Setup

The Kolmogorov Forward Equation governing the evolution of the IRFR is

$$\frac{\partial \varphi}{\partial t} = -\frac{\partial}{\partial x}[\mu(x,t,T)\varphi] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x,t,T)\varphi\right]. \tag{3.0.1}$$

Define the drift–diffusion pair

$$f(x,t,T) = -\mu(x,t,T), \qquad g^2(x,t,T) = \tfrac{1}{2}\sigma^2(x,t,T), \tag{3.0.2–3.0.3}$$

so that (3.0.1) becomes

$$\frac{\partial \varphi}{\partial t} = \frac{\partial^2}{\partial x^2}\left[g^2(x,t,T)\varphi\right] + \frac{\partial}{\partial x}[f(x,t,T)\varphi]. \tag{3.0.4}$$

3.1 Affine Forms of Drift and Diffusion

Statement 3.1 (Affine Forms of Drift and Diffusion)**.** Assume Axiom 2 (rolling–maturity invariance), so that the drift and diffusion depend on time only through the remaining maturity $\tau := T - t$:

$$f(x,t,T) = F(x,\tau), \qquad g(x,t,T) = G(x,\tau).$$

Assume in addition the following affine-closure hypothesis: for each fixed τ, the diffusion has state-independent slope,

$$\partial_{xx} G(x,\tau) = 0,$$

and the net probability flux is affine in the state variable, i.e. there exist functions $A(\tau)$ and $B(\tau)$ such that

$$\partial_x\big(G(x,\tau)^2\big) + F(x,\tau) = A(\tau)x + B(\tau).$$

Then both the diffusion and drift are affine in the IRFR state x:

$$g(x,t,T) = a(\tau)x + b(\tau), \qquad f(x,t,T) = a_1(\tau)x + b_1(\tau), \qquad \tau = T - t. \tag{3.1.1}$$

Derivation. Expanding the KFE (3.0.4) gives

$$\begin{aligned}\frac{\partial \varphi}{\partial t} &= g^2\varphi_{xx} + \big(2(g^2)_x + f\big)\varphi_x + \big((g^2)_{xx} + f_x\big)\varphi \\ &= g^2\varphi_{xx} + \big(4gg_x + f\big)\varphi_x + \big((g^2)_{xx} + f_x\big)\varphi. \end{aligned} \tag{3.1.2}$$

Under Axiom 2 we may write $f(x,t,T) = F(x,\tau)$ and $g(x,t,T) = G(x,\tau)$, where $\tau = T - t$. The additional affine-closure hypothesis now implies

$$\partial_{xx} G(x,\tau) = 0,$$

hence for each fixed τ,

$$G(x,\tau) = a(\tau)x + b(\tau).$$

Substituting this into the affine flux identity

$$\partial_x\big(G(x,\tau)^2\big) + F(x,\tau) = A(\tau)x + B(\tau)$$

yields

$$F(x,\tau) = A(\tau)x + B(\tau) - \partial_x\big(G(x,\tau)^2\big).$$

Since $G(x,\tau) = a(\tau)x + b(\tau)$, we have

$$\partial_x\big(G(x,\tau)^2\big) = 2a(\tau)\big(a(\tau)x + b(\tau)\big),$$

which is affine in x. Therefore $F(x,\tau)$ is affine in x as well:

$$F(x,\tau) = a_1(\tau)x + b_1(\tau).$$

Returning to the original notation,

$$g(x,t,T) = a(T-t)x + b(T-t), \qquad f(x,t,T) = a_1(T-t)x + b_1(T-t).$$

This proves Statement 3.1.

3.2 Equilibrium Relation of Drift and Diffusion

Statement 3.2 (Equilibrium Relation of Drift and Diffusion)**.** Let $\tau := T - t$. Suppose the drift and diffusion are affine in the IRFR state:

$$f(x,t,T) = a_1(\tau)x + b_1(\tau), \qquad g(x,t,T) = a(\tau)x + b(\tau).$$

Assume further that the equilibrium mean depends only on remaining maturity,

$$\mathbb{E}[x_{t,T}] = x_{e,\tau},$$

and that the affine intercepts satisfy the proportionality condition

$$b(\tau) = \frac{a(\tau)}{a_1(\tau)}\, b_1(\tau), \qquad a_1(\tau) \neq 0.$$

Then the drift and diffusion satisfy

$$f(x,t,T) = a_1(\tau)\big(x - x_{e,\tau}\big) + x'_{e,\tau}, \tag{3.2.1a}$$

$$g(x,t,T) = a(\tau)\big(x - x_{e,\tau}\big) + \frac{a(\tau)}{a_1(\tau)}\, x'_{e,\tau}. \tag{3.2.1b}$$

3.3 Deterministic Evolution of the IRFR Expectation

Statement 3.3 (Expected forward-curve dynamics)**.** For fixed maturity T, suppose the expected forward rate evolves according to

$$\frac{\partial \mathbb{E}[x_{t,T}]}{\partial t} = -a_1\big(\mathbb{E}[x_{t,T}] - x_{e,T-t}\big) + \frac{\partial x_{e,T-t}}{\partial t},$$

where $a_1 > 0$ is constant. Then

$$\mathbb{E}[x_{t,T}] = x_{e,T-t} + \big(x_{t_0,T} - x_{e,T-t_0}\big)\, e^{-a_1(t-t_0)}. \tag{3.3.1}$$

Derivation is given in Appendix A.3.

3.4 Baseline Drift–Diffusion Identification

Statement 3.4 (Baseline Drift–Diffusion Identification)**.** In the baseline specification, the affine drift loading is identified with the squared diffusion loading:

$$a_1 = a^2. \tag{3.4.1}$$

Reasoning. The relation $a_1 = a^2$ is imposed as the baseline structural identification linking temporal mean transport to variance production. The second-moment equation derived in Appendix A.4 shows that, before this identification is imposed, the variance dynamics contain an additional term proportional to $(a^2 - a_1)\,\mathrm{Var}(x_{t,T})$. Setting $a_1 = a^2$ removes this independent variance scale and yields the one-parameter baseline transport structure used throughout the remainder of the paper.

Interpretively, this is a drift–diffusion normalization: it ties the drift loading to the squared diffusion loading so that the transport speed of the mean and the variance-production scale are not treated as independent structural degrees of freedom in the baseline model.

3.5 Interim Drift and Diffusion Forms

Statement 3.5 (Interim Drift and Diffusion)**.** If one adopts the baseline exponential equilibrium specification

$$x_{e,T-t} = x_\infty + Ae^{-\lambda^2(T-t)},$$

then consistency of (3.2.1a)–(3.2.1b) implies the interim representations

$$f(x,t,T) = a^2 x_{t,T} - a^2 x_\infty - (\lambda^2 + a^2)A\, e^{-\lambda^2(T-t)}, \tag{3.5.1a}$$

$$g(x,t,T) = a x_{t,T} - a x_\infty - \left(a + \frac{\lambda^2}{a}\right) A\, e^{-\lambda^2(T-t)}. \tag{3.5.1b}$$

This is a conditional consistency result. The exponential equilibrium curve is imposed as a baseline specification, and (3.5.1a)– (3.5.1b) are the corresponding drift and diffusion forms required by that choice.

3.6 Instantaneous Variance

Statement 3.6 (Instantaneous Variance)**.**

$$\frac{\partial}{\partial t}\operatorname{Var}(x_{t,T}) = 2\left[a\left(x_{t_0,T} - x_\infty - Ae^{-\lambda^2(T-t_0)}\right)e^{-a^2(t-t_0)} - \frac{\lambda^2}{a}Ae^{-\lambda^2(T-t)}\right]^2.$$

Equivalently, defining

$$D(t,T) := \left(x_{t_0,T} - x_\infty - Ae^{-\lambda^2(T-t_0)}\right)e^{-a^2(t-t_0)},$$

this may be written compactly as

$$\frac{\partial}{\partial t}\operatorname{Var}(x_{t,T}) = 2\left(aD(t,T) - \frac{\lambda^2}{a}Ae^{-\lambda^2(T-t)}\right)^2.$$

.

A full derivation is given in Appendix A.6.

3.7 Market Signal–to–Noise Ratio

Statement 3.7 (Market Signal–to–Noise Ratio)**.** Define the Market Signal–to–Noise Ratio (MSNR) as the ratio of expected instantaneous return to the square root of instantaneous variance. Under this definition, the model-implied MSNR is

$$\mathrm{MSNR} = \frac{a}{\sqrt{2}}. \tag{3.7.1}$$

In particular, the MSNR is constant: it is independent of calendar time t, maturity T, and the state $x_{t,T}$. This does not require imposing the baseline specification $\lambda^2 = a^2$. A derivation is given in Appendix A.7.

4 Kolmogorov Forward Equation Solution

This section solves the Kolmogorov Forward Equation for the baseline model. In what follows, we impose

$$a_1 = a^2, \qquad \lambda^2 = a^2,$$

as structural baseline identifications and study the implied equilibrium behaviour of the IRFR density.

Interpretively, $a_1 = a^2$ identifies the temporal drift loading with the variance-production scale, while $\lambda^2 = a^2$ selects the structural branch in which maturity decay and calendar-time propagation share the same scale. The latter branch is justified in Appendix B.1: the Frobenius calculation first yields an inverse-gamma family indexed by $r = \lambda^2/a^2$, and the baseline model then selects the order-3 branch $r = 1$.

A convenient shifted coordinate removes the deterministic equilibrium component and isolates the residual stochastic part of the dynamics. Throughout the KFE solution, $x_{e,\tau}$

denotes the deterministic equilibrium curve, while $\widetilde{x}_{t,T}$ denotes the shifted stochastic state. In this coordinate, the KFE admits a closed-form equilibrium regime given by a shifted inverse-gamma distribution of order 3.

Throughout, we work with the maturity-indexed equilibrium regime, parameterised by

$$\tau := T - t \geq 0.$$

Accordingly, distributional results are stated as maturity-indexed equilibrium densities, while the limit $\tau \to \infty$ will be referred to as the *long-maturity limit.*

4.1 Maturity-Indexed Equilibrium Density

Under the baseline specification $\lambda^2 = a^2$, define the shifted variable

$$\widetilde{x}_{t,T} = x_{t,T} - x_\infty \left(1 - e^{-a^2(T-t)}\right).$$

This transformation removes the deterministic equilibrium component and centres the KFE on the residual stochastic part.

In the shifted coordinate, the forward equation becomes

$$\frac{\partial \varphi}{\partial t} = a^2 \widetilde{x}_{t,T}^2 \frac{\partial^2 \varphi}{\partial \widetilde{x}_{t,T}^2} + 5a^2 \widetilde{x}_{t,T} \frac{\partial \varphi}{\partial \widetilde{x}_{t,T}} + 3a^2 \varphi - a^2 x_\infty e^{-a^2(T-t)} \frac{\partial \varphi}{\partial \widetilde{x}_{t,T}}. \tag{4.1.1}$$

The Frobenius expansion in Appendix B.1 yields the equilibrium density

$$\varphi(\widetilde{x}_{t,T}, t, T) = \frac{x_\infty^3 e^{-3a^2(T-t)}}{\Gamma(3)\, \widetilde{x}_{t,T}^4} \exp\left(- \frac{x_\infty e^{-a^2(T-t)}}{\widetilde{x}_{t,T}} \right). \tag{4.1.2}$$

This is a shifted inverse-gamma law with shape parameter 3. Its support is strictly positive, and its heavy right tail reflects the multiplicative structure of the underlying diffusion. The scale is set by x_∞, while the factor $e^{-a^2(T-t)}$ controls the maturity-dependent approach to equilibrium.

4.2 Boundary Behaviour

We now classify the behaviour of the process near the lower boundary $x = 0$. For the diffusion associated with the IRFR, the scale density is

$$s'(x) = \exp\left(- \int^x \frac{2f(y,t,T)}{g^2(y,t,T)}\, dy \right) \sim x^{-2}, \qquad x \to 0^+, \tag{4.2.1}$$

and the corresponding speed density is

$$m(x) = \frac{1}{g^2(x,t,T)\, s'(x)} \sim x, \qquad x \to 0^+. \tag{4.2.2}$$

Hence

$$\int_{0^+} s'(x)\, dx = \infty, \qquad \int_{0^+} m(x)\, dx < \infty.$$

By the standard Feller boundary classification, the origin is therefore an *entrance boundary*. Equivalently,

$$\boxed{x = 0 \text{ is inaccessible from the interior and cannot be reached in finite time.}} \tag{4.2.3}$$

This is fully consistent with Axiom 4 and confirms that the model preserves positivity of the instantaneous short-maturity IRFR.

5 Integrated General r-Baseline Specification

This section consolidates the structural results established in Section 3 and Appendix A. We retain the baseline drift–diffusion identification

$$a_1 = a^2,$$

but we do not impose $\lambda^2 = a^2$. Instead, define the positive dimensionless ratio

$$r := \frac{\lambda^2}{a^2} > 0, \qquad \lambda^2 = ra^2. \tag{5.0.1}$$

The case $r = 1$ is therefore a special order-three branch, not a mathematical requirement for inverse-gamma solvability.

The purpose of the present section is to integrate the general r-branch into a single coherent specification for the equilibrium curve, the drift and diffusion coefficients, the Kolmogorov Forward Equation, the forward-curve representation, and the implied expectation and variance dynamics.

5.1 Boundary Condition and Determination of the Amplitude A

From Statement 3.5, the interim diffusion is

$$g(x, t, T) = ax_{t,T} - ax_\infty - \left(a + \frac{\lambda^2}{a}\right) Ae^{-\lambda^2(T-t)}.$$

Axiom 4 requires that the diffusion vanish at the short-maturity boundary:

$$g(x_{\min}, t, t) = 0.$$

At the boundary $x_{t,t} = x_{\min} = 0$, and at $T = t$ one has $e^{-\lambda^2(T-t)} = 1$. Hence

$$0 = -ax_\infty - \left(a + \frac{\lambda^2}{a}\right) A.$$

Using $\lambda^2 = ra^2$, this becomes

$$0 = -ax_\infty - a(1 + r)A.$$

Since $a \neq 0$, it follows that

$$A = -\frac{x_\infty}{1 + r}. \tag{5.1.1}$$

The previous value $A = -x_\infty/2$ is recovered only when $r = 1$.

5.2 General r-Equilibrium Curve

The exponential equilibrium form is

$$x_{e,T-t} = x_\infty + Ae^{-\lambda^2(T-t)}.$$

Substituting (5.1.1) and $\lambda^2 = ra^2$ gives

$$x_{e,T-t} = x_\infty \left(1 - \frac{1}{1+r} e^{-ra^2(T-t)}\right). \tag{5.2.1}$$

In particular,

$$x_{e,0} = \frac{r}{1+r} x_\infty. \tag{5.2.2}$$

Thus the equilibrium short end is not generally one half of the long-maturity regime level. The halfway result $x_{e,0} = x_\infty/2$ is the special case $r = 1$.

5.3 Updated Drift and Diffusion

Using $\lambda^2 = ra^2$ and $A = -x_\infty/(1+r)$ in the interim forms (3.5.1a)–(3.5.1b), we obtain

$$g(x_{t,T}, t, T) = a\left[x_{t,T} - x_\infty\left(1 - e^{-ra^2(T-t)}\right)\right], \tag{5.3.1a}$$

$$f(x_{t,T}, t, T) = a^2\left[x_{t,T} - x_\infty\left(1 - e^{-ra^2(T-t)}\right)\right]. \tag{5.3.1b}$$

It is useful to introduce the shifted coordinate

$$\widetilde{x}^{(r)}_{t,T} := x_{t,T} - x_\infty\left(1 - e^{-ra^2(T-t)}\right). \tag{5.3.2}$$

Then the drift–diffusion pair is simply

$$g = a\widetilde{x}^{(r)}_{t,T}, \qquad f = a^2\widetilde{x}^{(r)}_{t,T}.$$

The stochastic loading and the drift loading remain tied by $a_1 = a^2$, while the maturity-decay scale is governed by $ra^2 = \lambda^2$.

5.4 Updated Kolmogorov Forward Equation

Substituting (5.3.1a) and (5.3.1b) into the Kolmogorov Forward Equation gives

$$\begin{aligned}\frac{\partial\varphi}{\partial t} = \frac{\partial^2}{\partial x^2}&\left(a^2\left[x_{t,T} - x_\infty\left(1 - e^{-ra^2(T-t)}\right)\right]^2\varphi\right)\\ &+ \frac{\partial}{\partial x}\left(a^2\left[x_{t,T} - x_\infty\left(1 - e^{-ra^2(T-t)}\right)\right]\varphi\right).\end{aligned} \tag{5.4.1}$$

Equivalently, in the shifted coordinate $\widetilde{x}^{(r)}_{t,T}$, the coefficients are proportional to $\widetilde{x}^{(r)}_{t,T}$ itself. This is the integrated forward equation for the general r-branch.

5.5 Forward-Curve Representation

Using the exponential equilibrium curve together with the forward-curve representation from Section 3,

$$x_{t,T} = x_\infty + Ae^{-\lambda^2(T-t)} + \big(x_{t,t} - x_\infty - A\big)e^{-a^2(T-t)}.$$

Substituting $A = -x_\infty/(1+r)$ and $\lambda^2 = ra^2$ gives

$$x_{t,T} = x_\infty - \frac{x_\infty}{1+r}e^{-ra^2(T-t)} + \left(x_{t,t} - \frac{r}{1+r}x_\infty\right)e^{-a^2(T-t)}. \tag{5.5.1}$$

Thus the general r-branch produces a two-exponential forward-curve representation, even though the stochastic evolution remains one-factor. When $r = 1$, the two exponential loadings coincide and (5.5.1) collapses to

$$x_{t,T} = x_\infty + (x_{t,t} - x_\infty)e^{-a^2(T-t)}.$$

5.6 Expectation and Variance under the General r-Branch

For fixed maturity T, Statement 3.3 gives

$$\mathbb{E}[x_{t,T}] = x_{e,T-t} + (x_{t_0,T} - x_{e,T-t_0})\, e^{-a^2(t-t_0)}.$$

Using (5.5.1) at time t_0, this becomes

$$\mathbb{E}[x_{t,T}] = x_\infty - \frac{x_\infty}{1+r}e^{-ra^2(T-t)} + \left(x_{t_0,t_0} - \frac{r}{1+r}x_\infty\right)e^{-a^2(T+t-2t_0)}. \tag{5.6.1}$$

For the instantaneous variance, specialize Statement 3.6 using $\lambda^2 = ra^2$ and $A = -x_\infty/(1+r)$. This yields

$$\frac{\partial}{\partial t}\operatorname{Var}(x_{t,T}) = 2a^2\left[\left(x_{t_0,t_0} - \frac{r}{1+r}x_\infty\right)e^{-a^2(T+t-2t_0)} + \frac{r}{1+r}x_\infty e^{-ra^2(T-t)}\right]^2. \tag{5.6.2}$$

The expression inside the square is a two-exponential maturity function when $r \neq 1$. The following statement makes the one-factor volatility hump explicit.

Statement 5.1 (One-factor humped volatility)**.** *Fix calendar time t, set $\tau = T - t$, $k = a^2$, and $\Delta = t - t_0$. Define*

$$A_r := \left(x_{t_0,t_0} - \frac{r}{1+r}x_\infty\right)e^{-2k\Delta}, \qquad B_r := \frac{r}{1+r}x_\infty.$$

Then the IRFR local variance-rate can be written as

$$v_x(\tau) := \frac{\partial}{\partial t}\operatorname{Var}(x_{t,t+\tau}) = 2a^2\left(A_r e^{-k\tau} + B_r e^{-rk\tau}\right)^2. \tag{5.6.3}$$

Thus the maturity profile contains two deterministic loadings whenever $r \neq 1$. In particular, suppose $0 < r < 1$ and

$$rB_r < -A_r < B_r.$$

Then v_x has a strict interior maximum at

$$\tau_v^* = \frac{\log\left(-A_r/(rB_r)\right)}{(1-r)k}. \tag{5.6.4}$$

The covariance-rate kernel remains rank one.

Proof. Using $T = t + \tau$, equation (5.6.2) becomes

$$v_x(\tau) = 2a^2 H_r(\tau)^2, \qquad H_r(\tau) := A_r e^{-k\tau} + B_r e^{-rk\tau},$$

which proves (5.6.3). Under the stated conditions, $B_r > 0$, $A_r < 0$, and $H_r(0) = A_r + B_r > 0$. Moreover

$$H_r'(\tau) = -kA_r e^{-k\tau} - rkB_r e^{-rk\tau}.$$

The equation $H_r'(\tau) = 0$ is equivalent to

$$e^{-(1-r)k\tau} = -\frac{rB_r}{A_r},$$

and the stated inequalities imply a unique positive solution (5.6.4). For $\tau < \tau_v^*$, $H_r'(\tau) > 0$, while for $\tau > \tau_v^*$, $H_r'(\tau) < 0$. Since $H_r(\tau) > 0$, the derivative $v_x'(\tau) = 4a^2 H_r(\tau) H_r'(\tau)$ changes sign from positive to negative at τ_v^*, proving that the maximum is strict and interior. Finally, the model is driven by one Brownian shock; the local stochastic loading at maturity S is proportional to $H_r(S-t)$. Hence the instantaneous covariance-rate kernel has the product form

$$\frac{\partial}{\partial t}\operatorname{Cov}(x_{t,S_1}, x_{t,S_2}) = 2a^2 H_r(S_1 - t) H_r(S_2 - t),$$

which is rank one. □

The inequalities in Statement 5.1 are sufficient conditions identifying an open parameter region of the admissible general r-branch. They are not additional model axioms. The same two-loading mechanism is inherited by conventional rates through maturity averaging.

6 Economic Interpretation

This section interprets the structural and probabilistic results obtained in Sections 3–4, with the general ratio $r = \lambda^2/a^2$ retained. The parameter a controls the one-factor stochastic loading and the associated drift scale, while r separates the maturity-decay scale from the drift–diffusion scale.

6.1 Mean-Reversion Dynamics

The drift and diffusion coefficients in their updated form (5.3.1b)–(5.3.1a) may be rewritten as

$$f(x,t,T) = a^2(x - x_\infty) + a^2 x_\infty e^{-ra^2(T-t)}, \tag{6.1.1}$$

$$g(x,t,T) = a(x - x_\infty) + a x_\infty e^{-ra^2(T-t)}. \tag{6.1.2}$$

Thus, a sets the diffusion loading, a^2 governs the corresponding drift scale, and r determines the rate at which the maturity-dependent displacement decays toward the long-maturity regime level.

6.2 Equilibrium Regime Level and Equilibrium Structure

The equilibrium curve derived in (5.2.1) is

$$x_{e,T-t} = x_\infty - \frac{x_\infty}{1+r} e^{-ra^2(T-t)}, \tag{6.2.1}$$

which displays exponential convergence from the short end toward the equilibrium regime asymptote. At zero maturity,

$$x_{e,0} = \frac{r}{1+r} x_\infty.$$

Hence r controls the structural link between the short end and the long-maturity regime level. The earlier halfway relation is recovered only on the order-three branch $r = 1$.

6.3 Forward-Curve Dynamics and Hump Formation

Under the general r-branch, the expected forward curve evolves as

$$\mathbb{E}[x_{t,T}] = x_\infty - \frac{x_\infty}{1+r} e^{-ra^2(T-t)} + \left(x_{t_0,t_0} - \frac{r}{1+r} x_\infty \right) e^{-a^2(T+t-2t_0)}. \tag{6.3.1}$$

The same two exponential maturity loadings appear in the deterministic forward-curve representation (5.5.1):

$$e^{-a^2(T-t)} \qquad \text{and} \qquad e^{-ra^2(T-t)}.$$

Although the stochastic model is one-factor, the cross-sectional curve is therefore not forced to be monotone. This mechanism differs from earlier humped-volatility specifications in which the hump is introduced directly through the volatility function or through additional Markov state variables.

Let $\tau = T - t$. Differentiating (5.5.1) with respect to τ, an interior turning point satisfies

$$\frac{r x_\infty}{1+r} e^{-ra^2\tau} = \left(x_{t,t} - \frac{r}{1+r} x_\infty \right) e^{-a^2\tau}. \tag{6.3.2}$$

When the right-hand side is positive and the resulting maturity is positive, the turning point is located at

$$\tau^* = \frac{\log\left(\dfrac{(1+r)x_{t,t} - r x_\infty}{r x_\infty} \right)}{(1-r)a^2}. \tag{6.3.3}$$

Thus the general r-branch can generate monotone, humped, or inverted term structures within a one-factor specification. The special case $r = 1$ collapses the two maturity loadings into a single exponential and removes this hump-generating degree of freedom.

Covariance-style structure and equilibrium. Introducing shifted variables $T' = T - t_0$ and $t' = t - t_0$, so that $\tau = T' - t'$, separates elapsed-time decay from maturity loading. The transition structure remains reminiscent of covariance kernels in classical affine term-structure models such as Vasicek and CIR, while $r \neq 1$ permits two distinct deterministic maturity scales.

Relation to stationary one-factor restrictions. The admissibility of humped volatility in the present framework should not be read as contradicting standard results which preclude humped volatility in single-factor stationary term-structure models. Those results apply to stationary volatility specifications, or to models in which the maturity-loading structure is fixed by the stationary equilibrium. The humped volatility obtained here is not asserted to be a stationary-equilibrium object. Rather, it arises along the non-stationary rolling-maturity branch when the temporal drift-diffusion scale and the maturity-decay scale are allowed to differ. Indeed, Axiom 3 imposes equilibrium transport only on the first two moments; it does not, by itself, require or exclude analogous transport of higher-order moments. In the solved KFE equilibrium regime, however, the full maturity-indexed density is obtained explicitly as a shifted inverse-gamma law. Consequently, all finite moments admitted by that density inherit the same maturity-indexed equilibrium structure. The point relevant for the present section is that this equilibrium regime has a monotone instantaneous volatility loading in maturity and therefore does not generate a stationary humped volatility curve.

The contribution of the present section is instead to show that humped volatility is admissible within a structurally consistent one-factor framework outside the monotone equilibrium benchmark, through the interaction of the two deterministic maturity loadings induced by $r \neq 1$.

This distinction is important. The model does not evade stationary one-factor restrictions by imposing an exogenous humped volatility function. Nor does it introduce an additional Brownian factor. It shows that, once rolling-maturity invariance is imposed before stationarity is specialised, the admissible one-factor family contains non-stationary term structures whose conventional-rate volatility may be humped. The stationary equilibrium remains monotone; the humped profile is an admissible transitional or off-equilibrium configuration.

6.4 Admissible Yield-Curve Shapes

The preceding discussion shows that humped volatility term structures are admissible within the one-factor rolling-rate framework. The same two-scale structure also permits a richer class of deterministic yield-curve shapes than the monotone equilibrium benchmark might suggest.

Let

$$\Delta = x_{t_0,t_0} - x_\infty, \qquad \lambda^2 = ra^2,$$

and consider the stylised two-loading initial rolling forward curve

$$x(t_0, t_0 + \tau) = x_\infty + Ae^{-a^2\tau} + Be^{-ra^2\tau},$$

where

$$A + B = \Delta.$$

The associated conventional continuously compounded yield is

$$y(t_0, t_0+\tau) = x_\infty + A\frac{1-e^{-a^2\tau}}{a^2\tau} + B\frac{1-e^{-ra^2\tau}}{ra^2\tau}.$$

This representation separates the level restriction,

$$y(t_0, t_0) = x_{t_0,t_0},$$

from the shape restriction. The parameters a^2 and r determine the maturity scales, while the loading split between A and B determines whether the initial curve is monotone, humped, or locally dipped.

The short-end slope of the conventional yield curve is obtained from the expansion

$$\frac{1-e^{-k\tau}}{k\tau} = 1 - \frac{k\tau}{2} + O(\tau^2),$$

which gives

$$\left.\frac{\partial y(t_0, t_0+\tau)}{\partial \tau}\right|_{\tau=0} = -\frac{a^2}{2}(A + rB).$$

Using $A = \Delta - B$, this becomes

$$\left.\frac{\partial y(t_0, t_0+\tau)}{\partial \tau}\right|_{\tau=0} = -\frac{a^2}{2}\left[\Delta + (r-1)B\right].$$

Hence the conventional yield curve initially slopes upward if

$$\Delta + (r-1)B < 0,$$

and initially slopes downward if

$$\Delta + (r-1)B > 0.$$

The long-maturity behaviour is governed by

$$y(t_0, t_0+\tau) - x_\infty \sim \frac{1}{a^2\tau}\left(A + \frac{B}{r}\right) \qquad \text{as } \tau \to \infty.$$

Thus the sign of

$$A + \frac{B}{r} = \Delta - \left(1 - \frac{1}{r}\right)B$$

determines whether the conventional yield approaches the long-maturity level from above or below.

A sufficient condition for an initially upward-sloping yield curve to possess an interior hump is therefore

$$\Delta + (r-1)B < 0$$

together with

$$\Delta - \left(1 - \frac{1}{r}\right)B > 0.$$

The first condition makes the curve rise away from the short end. The second condition makes the long end approach x_∞ from above, so that the curve must eventually turn down. Under the usual two-loading cases considered here, this gives a single interior maximum.

For $r > 1$, these inequalities may be written as

$$B < -\frac{\Delta}{r-1},$$

and

$$B < \frac{r\Delta}{r-1}.$$

The admissible interval depends on the sign of $\Delta = x_{t_0,t_0} - x_\infty$. Thus humped yield curves may occur both when the short rate lies below the long-maturity level and when it lies above it, provided the two maturity loadings have opposite signs of sufficient magnitude.

For example, take

$$x_{t_0,t_0} = 2.00\%, \qquad x_\infty = 3.00\%, \qquad a^2 = 0.25, \qquad r = 4.$$

Then $\lambda^2 = 1.00$. Choosing

$$B = -2.50\%, \qquad A = 1.50\%,$$

gives

$$A + B = -1.00\% = x_{t_0,t_0} - x_\infty.$$

The initial yield curve slopes upward, rises above x_∞, and then declines back toward the long-maturity level. This is a conventional humped yield curve generated entirely by the two deterministic maturity loadings.

A second example is obtained with

$$x_{t_0,t_0} = 3.50\%, \qquad x_\infty = 2.50\%, \qquad a^2 = 0.25, \qquad r = 4,$$

and

$$B = -2.00\%, \qquad A = 3.00\%.$$

Here the short rate already lies above the long-maturity level, but the yield curve can still rise initially before turning down. Hence a humped yield curve is not restricted to cases in which the short rate begins below the long end.

By contrast, if the short-end slope is negative,

$$\Delta + (r-1)B > 0,$$

the same two-loading structure does not generally produce a single humped conventional yield curve. It may produce a monotone downward curve, or a local trough followed by convergence, depending on the sign of

$$\Delta - \left(1 - \frac{1}{r}\right) B.$$

Thus a yield curve that is already falling at the short end and then later develops a conventional interior hump would require additional structure beyond the bare two-loading specification.

This is economically natural: such behaviour resembles an externally disturbed curve rather than the undistorted rolling-maturity equilibrium.

The examples above show that the framework has useful flexibility for fitting observed term-structure shapes. However, we would caution against interpreting the present specification as a complete empirical model of observed yield curves. In practice, central-bank policy, administered rates, quantitative easing, liquidity regulation, collateral scarcity, and other institutional interventions can impose local distortions on the short and intermediate parts of the curve. For empirical testing and practitioner calibration, the natural next step is therefore to extend the framework to include a policy-intervention or exogenous-distortion component. The present model should be viewed as the structurally admissible one-factor benchmark against which such interventions can be measured.

6.5 Distributional Implications

The Frobenius calculation in Appendix B.1 yields an inverse-gamma family indexed by $r = \lambda^2/a^2$. In the shifted coordinate (5.3.2), the maturity-indexed equilibrium density is

$$\varphi(\widetilde{x}^{(r)}_{t,T}, t, T) = \frac{\left(r x_\infty e^{-ra^2(T-t)}\right)^{r+2}}{\Gamma(r+2)\left(\widetilde{x}^{(r)}_{t,T}\right)^{r+3}} \exp\left(-\frac{r x_\infty e^{-ra^2(T-t)}}{\widetilde{x}^{(r)}_{t,T}}\right), \qquad \widetilde{x}^{(r)}_{t,T} > 0. \tag{6.5.1}$$

Equivalently,

$$\widetilde{x}^{(r)}_{t,T} \sim \text{InvGamma}\left(r+2,\; r x_\infty e^{-ra^2(T-t)}\right).$$

The branch $r = 1$ has shape parameter three and recovers the order-three shifted inverse-gamma density displayed in Section 4. For general $r > 0$, the density remains strictly positive on its shifted support and has heavy right tail $(\widetilde{x}^{(r)}_{t,T})^{-(r+3)}$.

6.6 Boundary Behaviour

The lower boundary is most naturally expressed in the shifted coordinate $\widetilde{x}^{(r)}_{t,T}$. The boundary $\widetilde{x}^{(r)}_{t,T} = 0$ corresponds in the original IRFR state to

$$x_{t,T} = x_\infty \left(1 - e^{-ra^2(T-t)}\right).$$

This shifted boundary is inaccessible from the interior. At zero maturity, it reduces to the short-end boundary $x_{t,t} = 0$, consistent with Axiom 4.

7 Applications

The structural and probabilistic characterisation of the IRFR provides a tractable platform for simulation, estimation, curve construction, risk analysis, and pricing of interest-rate–dependent instruments. The general r-branch adds a further practical benefit: it allows a one-factor model to generate non-monotone term structures through two deterministic maturity loadings.

7.1 Simulating IRFR Paths

Using the updated drift and diffusion (5.3.1b)–(5.3.1a), the IRFR under the physical measure satisfies the stochastic differential equation

$$dx_{t,T} = -f(x,t,T)\,dt + \sqrt{2}\,g(x,t,T)\,dW_t.$$

In the shifted coordinate (5.3.2), this becomes a multiplicative diffusion around the maturity-dependent floor:

$$dx_{t,T} = -a^2\widetilde{x}^{(r)}_{t,T}\,dt + \sqrt{2}\,a\widetilde{x}^{(r)}_{t,T}\,dW_t.$$

The affine structure supports stable numerical implementation, while the shifted-boundary formulation gives a transparent positivity condition.

7.2 Yield Curve Construction

Under the general r-branch, the forward curve can be constructed from the short-maturity value $x_{t,t}$ using

$$x_{t,T} = x_\infty - \frac{x_\infty}{1+r}e^{-ra^2(T-t)} + \left(x_{t,t} - \frac{r}{1+r}x_\infty\right)e^{-a^2(T-t)}. \tag{7.2.1}$$

This representation is still parsimonious, but it is richer than the single-exponential $r = 1$ branch. In particular, it can generate upward-sloping, downward-sloping, and humped curves depending on the short-end displacement and the value of r.

7.3 Numerical Illustration: Conventional Rates with Humped Local Volatility

The same two-loading mechanism is inherited by conventional continuously compounded rates through the maturity average in Appendix C. The following stylised example is intended to illustrate the mechanism rather than to calibrate the model. Set

$$x_\infty = 4.00\%, \qquad x_{t_0,t_0} = 0.50\%, \qquad a = 0.70, \qquad a^2 = 0.49, \qquad r = 0.30, \qquad t = t_0.$$

Using (C.2.5), the conventional yield curve is upward sloping and non-inverted. The front-end yield is approximately 0.50%, the two-year yield is approximately 1.06%, the ten-year yield is approximately 2.30%, and the thirty-year yield is approximately 3.28%.

Using (C.3.7), the associated conventional-rate instantaneous volatility is non-zero for finite maturity and has an interior maximum close to the two-year sector. In this example the maximum occurs at approximately 1.96 years. The volatility hump is produced without adding a second stochastic factor: the covariance-rate kernel remains rank one, while the deterministic maturity profile contains two distinct loadings, $e^{-a^2\tau}$ and $e^{-ra^2\tau}$.

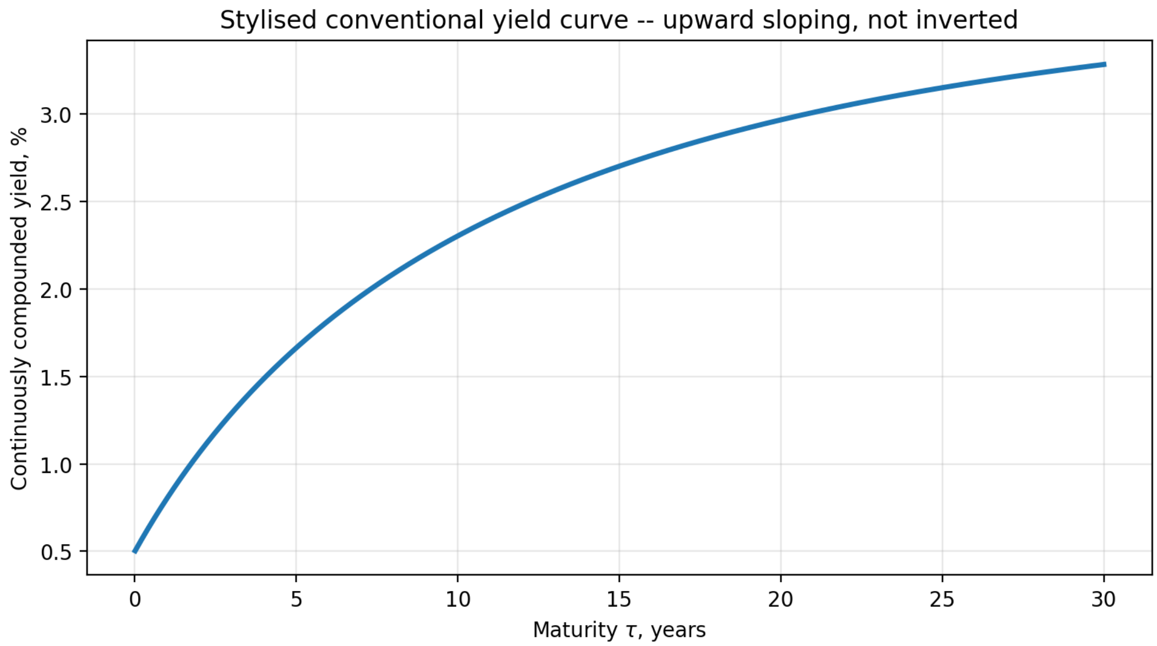

Figure 1: Stylised upward-sloping conventional yield curve under the general r-branch. The parameter set is $x_\infty = 4.00\%$, $x_{t_0,t_0} = 0.50\%$, $a = 0.70$, and $r = 0.30$, with $t = t_0$. The curve is upward sloping and non-inverted.

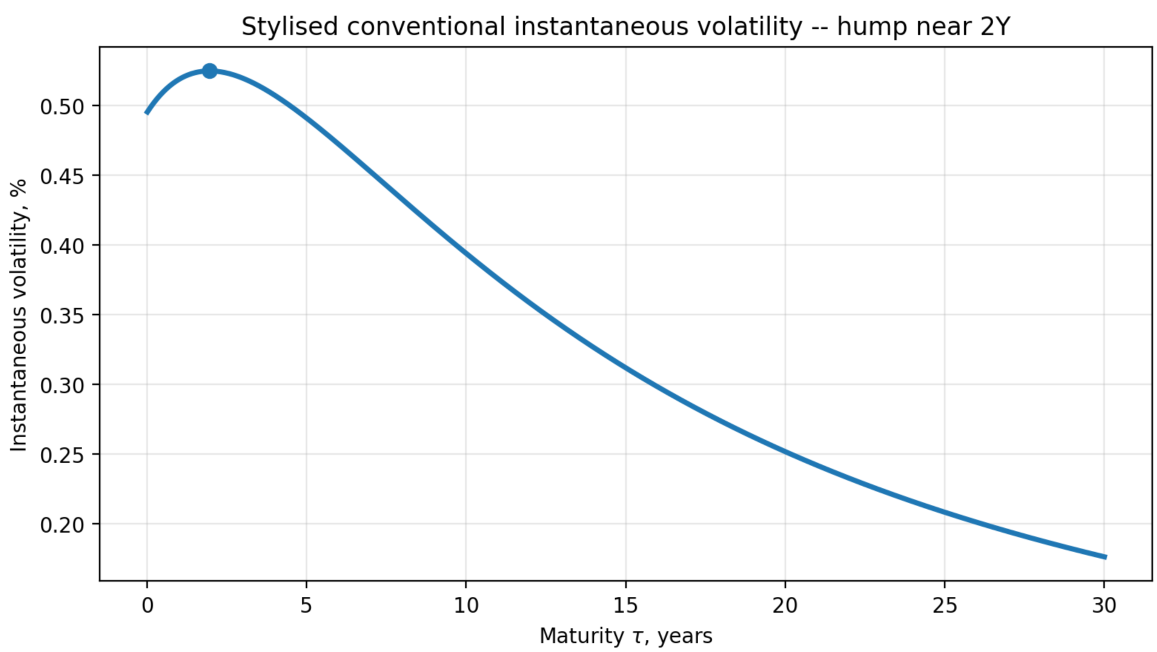

Figure 2: Stylised conventional-rate instantaneous volatility under the same parameter set as Figure 1. The volatility is non-zero for finite maturity and has an interior hump close to the two-year maturity, despite the underlying conventional yield curve being upward sloping rather than inverted.

7.4 Risk Management and Stress Testing

Because the maturity-indexed equilibrium density is heavy-tailed and strictly positive on its shifted support, stress scenarios may incorporate large rate moves drawn from the generalized shifted inverse-gamma law. Perturbations in (a, r, x_∞) provide a coherent way to stress-test mean reversion, maturity decay, equilibrium regime assumptions, and hump risk. The variance-rate formula (5.6.2) also identifies maturities at which local uncertainty is amplified by the interaction of the two exponential loadings.

7.5 Pricing Interest-Rate–Dependent Claims

Under a risk-neutral measure, the drift adjusts to incorporate market prices of risk, while the diffusion remains affine in the shifted state. The shifted positivity structure makes the model suitable for pricing caps, floors, swaptions, and other interest-rate–dependent claims. The two-exponential forward-curve representation gives a tractable basis for semi-closed-form pricing and calibration while preserving one-factor stochastic dynamics.

Appendix A Technical Derivations for Section 3

This appendix provides full derivations of the structural results in Section 3. All proofs follow a clear step-by-step format.

A.1 Proof of Statement 3.1

Step 1: Expand the KFE. From (3.0.4),

$$\frac{\partial \varphi}{\partial t} = g^2 \varphi_{xx} + \big(2(g^2)_x + f\big)\varphi_x + \big((g^2)_{xx} + f_x\big)\varphi. \tag{A.1.1}$$

Equivalently, since $(g^2)_x = 2gg_x$,

$$\frac{\partial \varphi}{\partial t} = g^2 \varphi_{xx} + (4gg_x + f)\varphi_x + \big((g^2)_{xx} + f_x\big)\varphi.$$

Step 2: Apply rolling–maturity invariance (Axiom 2). Axiom 2 implies that the coefficients depend on calendar time only through remaining maturity $\tau = T - t$. Hence there exist functions F and G such that

$$f(x, t, T) = F(x, \tau), \qquad g(x, t, T) = G(x, \tau), \qquad \tau = T - t. \tag{A.1.2}$$

Step 3: Impose the affine-closure hypothesis. Assume that for each fixed τ,

$$\partial_{xx} G(x, \tau) = 0, \tag{A.1.3}$$

and that the net probability flux is affine in x:

$$\partial_x\big(G(x, \tau)^2\big) + F(x, \tau) = A(\tau)x + B(\tau) \tag{A.1.4}$$

for some functions $A(\tau)$ and $B(\tau)$.

Step 4: Solve for the diffusion. Equation (A.1.3) implies that G is affine in x. Therefore there exist functions $a(\tau)$ and $b(\tau)$ such that

$$G(x, \tau) = a(\tau)x + b(\tau). \tag{A.1.5}$$

Step 5: Solve for the drift. Substituting (A.1.5) into (A.1.4) gives

$$F(x, \tau) = A(\tau)x + B(\tau) - \partial_x\big(G(x, \tau)^2\big).$$

But

$$\partial_x\big(G(x, \tau)^2\big) = 2a(\tau)\big(a(\tau)x + b(\tau)\big),$$

which is affine in x. Hence F is affine in x as well, so there exist functions $a_1(\tau)$ and $b_1(\tau)$ such that

$$F(x, \tau) = a_1(\tau)x + b_1(\tau). \tag{A.1.6}$$

Step 6: Return to the original notation. Using $\tau = T - t$, equations (A.1.5) and (A.1.6) become

$$g(x, t, T) = a(T - t)x + b(T - t), \qquad f(x, t, T) = a_1(T - t)x + b_1(T - t).$$

Thus both diffusion and drift are affine in the IRFR state x, which establishes Statement 3.1.

A.2 Proof of Statement 3.2

Goal. Derive the equilibrium drift–diffusion relations (3.2.1a)–(3.2.1b).

Step 1: Equilibrium identity. Under Axiom 3, the equilibrium IRFR depends only on time-to-maturity, so that

$$x_{e,T-t} = x_{e,\tau}, \qquad \tau := T - t.$$

Hence

$$\frac{\partial x_{e,T-t}}{\partial t} = -\, x'_{e,\tau}. \tag{A.2.12}$$

Step 2: First-moment evolution. From the Fokker–Planck equation, multiplying by x, integrating over $x \in [x_{\min}, \infty)$, and assuming the boundary terms vanish, gives

$$\frac{\partial}{\partial t}\mathbb{E}[x_{t,T}] = -\mathbb{E}[f(x, t, T)]. \tag{A.2.13}$$

Using the affine drift form from Statement 3.1,

$$f(x, t, T) = a_1(\tau)x + b_1(\tau), \tag{A.2.14}$$

it follows that

$$\frac{\partial}{\partial t}\mathbb{E}[x_{t,T}] = -\, a_1(\tau)\mathbb{E}[x_{t,T}] - b_1(\tau). \tag{A.2.15}$$

Step 3: Impose equilibrium. At equilibrium,

$$\mathbb{E}[x_{t,T}] = x_{e,\tau},$$

so (A.2.15) becomes

$$\frac{\partial x_{e,\tau}}{\partial t} = -\,a_1(\tau)x_{e,\tau} - b_1(\tau). \tag{A.2.16}$$

Using $\partial_t x_{e,\tau} = -x'_{e,\tau}$, we obtain

$$-\,x'_{e,\tau} = -\,a_1(\tau)x_{e,\tau} - b_1(\tau),$$

hence

$$b_1(\tau) = -\,a_1(\tau)x_{e,\tau} + x'_{e,\tau}. \tag{A.2.17}$$

Step 4: Drift relation. Substituting (A.2.17) into (A.2.14) yields

$$f(x,t,T) = a_1(\tau)x - a_1(\tau)x_{e,\tau} + x'_{e,\tau},$$

that is,

$$f(x,t,T) = a_1(\tau)\big(x - x_{e,\tau}\big) + x'_{e,\tau}. \tag{A.2.18}$$

Identifying x with the state variable $x_{t,T}$ gives (3.2.1a).

Step 5: Diffusion relation. Using the affine diffusion form from Statement 3.1,

$$g(x,t,T) = a(\tau)x + b(\tau), \tag{A.2.19}$$

together with the proportionality hypothesis

$$b(\tau) = \frac{a(\tau)}{a_1(\tau)}\,b_1(\tau), \tag{A.2.20}$$

and substituting (A.2.17), one finds

$$b(\tau) = \frac{a(\tau)}{a_1(\tau)}\Big(-\,a_1(\tau)x_{e,\tau} + x'_{e,\tau}\Big) = -\,a(\tau)x_{e,\tau} + \frac{a(\tau)}{a_1(\tau)}x'_{e,\tau}. \tag{A.2.21}$$

Therefore

$$g(x,t,T) = a(\tau)x - a(\tau)x_{e,\tau} + \frac{a(\tau)}{a_1(\tau)}x'_{e,\tau},$$

i.e.

$$g(x,t,T) = a(\tau)\big(x - x_{e,\tau}\big) + \frac{a(\tau)}{a_1(\tau)}x'_{e,\tau}. \tag{A.2.22}$$

Identifying x with the state variable $x_{t,T}$ gives (3.2.1b).

This proves Statement 3.2.

A.3 Proof of Statement 3.3

Goal. Show that, for fixed maturity T,

$$\mathbb{E}[x_{t,T}] = x_{e,T-t} + \left(x_{t_0,T} - x_{e,T-t_0}\right) e^{-a_1(t-t_0)}.$$

Step 1: Start from the expectation dynamics. By Statement 3.3, the expected forward rate satisfies

$$\frac{\partial \mathbb{E}[x_{t,T}]}{\partial t} = -a_1\left(\mathbb{E}[x_{t,T}] - x_{e,T-t}\right) + \frac{\partial x_{e,T-t}}{\partial t},$$

where $a_1 > 0$ is constant.

Step 2: Define the deviation from equilibrium. Let

$$z(t) := \mathbb{E}[x_{t,T}] - x_{e,T-t}.$$

Then

$$\frac{dz(t)}{dt} = \frac{\partial \mathbb{E}[x_{t,T}]}{\partial t} - \frac{\partial x_{e,T-t}}{\partial t}.$$

Substituting the expectation dynamics gives

$$\frac{dz(t)}{dt} = -a_1\left(\mathbb{E}[x_{t,T}] - x_{e,T-t}\right) = -a_1 z(t).$$

Step 3: Solve the differential equation. The solution is

$$z(t) = z(t_0)e^{-a_1(t-t_0)}.$$

Hence

$$\mathbb{E}[x_{t,T}] - x_{e,T-t} = \left(\mathbb{E}[x_{t_0,T}] - x_{e,T-t_0}\right) e^{-a_1(t-t_0)}.$$

Step 4: Apply the initial condition. At time t_0, the forward rate is known, so

$$\mathbb{E}[x_{t_0,T}] = x_{t_0,T}.$$

Therefore

$$\mathbb{E}[x_{t,T}] - x_{e,T-t} = \left(x_{t_0,T} - x_{e,T-t_0}\right) e^{-a_1(t-t_0)}.$$

Rearranging yields

$$\mathbb{E}[x_{t,T}] = x_{e,T-t} + \left(x_{t_0,T} - x_{e,T-t_0}\right) e^{-a_1(t-t_0)}. \tag{A.3.1}$$

This proves Statement 3.3.

A.4 Derivation of the Baseline Drift–Diffusion Identification

Goal. Derive the variance equation that motivates the baseline identification

$$a_1 = a^2.$$

The calculation below shows exactly where an independent drift scale would enter the second-moment dynamics.

Step 1: Start from the variance dynamics. Let

$$V(t,T) := \mathrm{Var}(x_{t,T}).$$

From the affine diffusion specification, the forward rate satisfies

$$dx_{t,T} = \big(-a_1 x_{t,T} + a_1 x_{e,T-t} + \partial_t x_{e,T-t}\big)\,dt + \big(a x_{t,T} + b(T-t)\big)\,dW_t,$$

where, by Statement 3.2,

$$b(T-t) = -a\,x_{e,T-t} + \frac{1}{a} x'_{e,T-t}.$$

Step 2: Derive the second-moment equation. Applying Itô's formula to $x_{t,T}^2$ and taking expectations gives

$$\begin{aligned}\frac{\partial}{\partial t}\mathbb{E}[x_{t,T}^2] &= 2\,\mathbb{E}\big[x_{t,T}\big(-a_1 x_{t,T} + a_1 x_{e,T-t} + \partial_t x_{e,T-t}\big)\big] + \mathbb{E}\Big[\big(a x_{t,T} + b(T-t)\big)^2\Big] \\ &= (-2a_1 + a^2)\mathbb{E}[x_{t,T}^2] + \big(2a_1 x_{e,T-t} + 2\partial_t x_{e,T-t} + 2ab(T-t)\big)\mathbb{E}[x_{t,T}] + b(T-t)^2.\end{aligned} \tag{A.4.1}$$

Step 3: Pass from the second moment to the variance. Since

$$V(t,T) = \mathbb{E}[x_{t,T}^2] - \mathbb{E}[x_{t,T}]^2,$$

differentiate with respect to t:

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial t}\mathbb{E}[x_{t,T}^2] - 2\,\mathbb{E}[x_{t,T}]\frac{\partial}{\partial t}\mathbb{E}[x_{t,T}].$$

Using Statement 3.3,

$$\frac{\partial}{\partial t}\mathbb{E}[x_{t,T}] = -a_1\big(\mathbb{E}[x_{t,T}] - x_{e,T-t}\big) + \partial_t x_{e,T-t}, \tag{A.4.2}$$

and substituting (A.4.1) and (A.4.2), the mixed terms cancel. This yields

$$\frac{\partial V}{\partial t} = 2(a^2 - a_1)V + \big(a\,x_{e,T-t} + b(T-t)\big)^2. \tag{A.4.3}$$

Step 4: Use the equilibrium diffusion relation. From Statement 3.2,

$$b(T-t) = -a\,x_{e,T-t} + \frac{1}{a} x'_{e,T-t}.$$

Hence

$$a\,x_{e,T-t} + b(T-t) = \frac{1}{a} x'_{e,T-t},$$

so (A.4.3) becomes

$$\frac{\partial V}{\partial t} = 2(a^2 - a_1)V + \frac{1}{a^2}\big(x'_{e,T-t}\big)^2. \tag{A.4.4}$$

Step 5: Identify the baseline one-scale case. Under equilibrium transport, admissible term-structure objects depend on calendar time and maturity through

$$\tau := T - t.$$

If the variance is written as $V(t,T) = \widetilde{V}(\tau)$, then

$$\frac{\partial V}{\partial t} = -\widetilde{V}'(\tau),$$

and (A.4.4) becomes

$$-\widetilde{V}'(\tau) = 2(a^2 - a_1)\widetilde{V}(\tau) + \frac{1}{a^2}\big(x'_e(\tau)\big)^2. \tag{A.4.5}$$

Equation (A.4.5) displays the structural choice clearly. If $a_1 \neq a^2$, the variance equation carries an additional homogeneous component proportional to $\widetilde{V}(\tau)$, introducing a second decay scale beyond the diffusion scale a. The baseline model rules out this additional independent variance scale by imposing the one-scale identification

$$a_1 = a^2. \tag{A.4.6}$$

Under this baseline identification, (A.4.4) reduces to

$$\frac{\partial V}{\partial t} = \frac{1}{a^2}\big(x'_{e,T-t}\big)^2,$$

so the variance production is governed by the same structural scale used in the drift–diffusion specification.

This establishes the baseline identification used in Statement 3.4.

A.5 Proof of Statement 3.5

Goal. Show that once the baseline exponential equilibrium specification is imposed, consistency of the equilibrium relations determines the interim forms (3.5.1a)–(3.5.1b).

Step 1: Start from the equilibrium drift and diffusion relations.

From Appendix A.2, the equilibrium relations are

$$f(x,t,T) = a_1 x_{t,T} - a_1 x_{e,T-t} - \frac{\partial x_{e,T-t}}{\partial t}, \tag{A.5.1a}$$

$$g(x,t,T) = a x_{t,T} - a x_{e,T-t} - \frac{a}{a_1}\frac{\partial x_{e,T-t}}{\partial t}. \tag{A.5.1b}$$

Since $\tau = T - t$, we have

$$-\frac{\partial x_{e,T-t}}{\partial t} = \frac{\partial x_{e,T-t}}{\partial T}.$$

Using the baseline identification $a_1 = a^2$, this becomes

$$f(x,t,T) = a^2 x_{t,T} - a^2 x_{e,T-t} + \frac{\partial x_{e,T-t}}{\partial T}, \tag{A.5.2a}$$

$$g(x,t,T) = a x_{t,T} - a x_{e,T-t} + \frac{1}{a}\frac{\partial x_{e,T-t}}{\partial T}. \tag{A.5.2b}$$

Step 2: Impose the baseline exponential equilibrium specification.

At this stage, no unique functional form for $x_{e,T-t}$ has been derived. We now impose the baseline exponential equilibrium specification used in the main text:

$$x_{e,T-t} = x_\infty + Ae^{-\lambda^2(T-t)}. \tag{A.5.3}$$

Equivalently, in the maturity variable $\tau = T - t$,

$$x_{e,\tau} = x_\infty + Ae^{-\lambda^2\tau}. \tag{A.5.4}$$

Differentiating (A.5.3) with respect to T yields

$$\frac{\partial x_{e,T-t}}{\partial T} = -\lambda^2 Ae^{-\lambda^2(T-t)}. \tag{A.5.5}$$

Step 3: Substitute into the equilibrium relations.

Substituting (A.5.3) and (A.5.5) into (A.5.2a) gives

$$\begin{aligned} f(x,t,T) &= a^2 x_{t,T} - a^2\Big(x_\infty + Ae^{-\lambda^2(T-t)}\Big) - \lambda^2 Ae^{-\lambda^2(T-t)} \\ &= a^2 x_{t,T} - a^2 x_\infty - (\lambda^2 + a^2)Ae^{-\lambda^2(T-t)}. \end{aligned} \tag{A.5.6}$$

Likewise, substituting (A.5.3) and (A.5.5) into (A.5.2b) gives

$$\begin{aligned} g(x,t,T) &= a x_{t,T} - a\Big(x_\infty + Ae^{-\lambda^2(T-t)}\Big) - \frac{\lambda^2}{a}Ae^{-\lambda^2(T-t)} \\ &= a x_{t,T} - a x_\infty - \left(a + \frac{\lambda^2}{a}\right)Ae^{-\lambda^2(T-t)}. \end{aligned} \tag{A.5.7}$$

Conclusion.

Thus, once the baseline exponential equilibrium curve is imposed, consistency of the equilibrium drift and diffusion relations yields the interim forms (3.5.1a)–(3.5.1b). This is a conditional identification result, not a uniqueness statement about all admissible equilibrium curves.

A.6 Proof of Statement 3.6

Goal. Derive the instantaneous variance-rate implied by the interim drift and diffusion forms obtained under the baseline exponential equilibrium specification.

Step 1: Moment identities from the KFE.

The Kolmogorov Forward Equation is

$$\frac{\partial \varphi}{\partial t} = \frac{\partial^2}{\partial x^2}\left(g^2(x,t,T)\varphi\right) + \frac{\partial}{\partial x}\left(f(x,t,T)\varphi\right).$$

Assuming the boundary terms vanish, multiplication by x and integration over the state space gives

$$\frac{\partial}{\partial t}\mathbb{E}[x_{t,T}] = -\mathbb{E}[f(x_{t,T},t,T)].$$

Likewise, multiplication by x^2 and integration by parts gives

$$\frac{\partial}{\partial t}\mathbb{E}[x_{t,T}^2] = -2\mathbb{E}[x_{t,T}f(x_{t,T},t,T)] + 2\mathbb{E}[g^2(x_{t,T},t,T)].$$

Step 2: Pass from second moments to variance.

Let

$$m(t,T) := \mathbb{E}[x_{t,T}], \qquad V(t,T) := \mathrm{Var}(x_{t,T}).$$

Since

$$V(t,T) = \mathbb{E}[x_{t,T}^2] - m(t,T)^2,$$

we obtain

$$\frac{\partial V}{\partial t} = -2\mathbb{E}[x_{t,T}f(x_{t,T},t,T)] + 2\mathbb{E}[g^2(x_{t,T},t,T)] + 2m(t,T)\mathbb{E}[f(x_{t,T},t,T)].$$

Step 3: Use the affine forms of f and g.

Under the interim baseline specification,

$$f(x,t,T) = a^2x - a^2x_\infty - (\lambda^2 + a^2)Ae^{-\lambda^2(T-t)},$$

and

$$g(x,t,T) = ax - ax_\infty - \left(a + \frac{\lambda^2}{a}\right)Ae^{-\lambda^2(T-t)}.$$

Write these as

$$f(x,t,T) = a^2x + \beta(t,T), \qquad g(x,t,T) = ax + \gamma(t,T),$$

where

$$\beta(t,T) = -a^2x_\infty - (\lambda^2 + a^2)Ae^{-\lambda^2(T-t)},$$

and

$$\gamma(t,T) = -ax_\infty - \left(a + \frac{\lambda^2}{a}\right)Ae^{-\lambda^2(T-t)}.$$

Since f is affine,

$$\mathbb{E}[f(x_{t,T},t,T)] = a^2m(t,T) + \beta(t,T),$$

and

$$\mathbb{E}[x_{t,T} f(x_{t,T}, t, T)] = a^2 \mathbb{E}[x_{t,T}^2] + \beta(t, T) m(t, T).$$

Therefore,

$$-2\mathbb{E}[x_{t,T} f(x_{t,T}, t, T)] + 2m(t, T)\mathbb{E}[f(x_{t,T}, t, T)] = -2a^2 V(t, T).$$

For the diffusion term,

$$\mathbb{E}[g^2(x_{t,T}, t, T)] = \mathbb{E}[(ax_{t,T} + \gamma(t, T))^2].$$

Hence

$$\mathbb{E}[g^2(x_{t,T}, t, T)] = a^2 V(t, T) + (am(t, T) + \gamma(t, T))^2 .$$

Substituting into the variance equation gives

$$\frac{\partial V}{\partial t} = -2a^2 V(t, T) + 2a^2 V(t, T) + 2 (am(t, T) + \gamma(t, T))^2 .$$

The variance terms cancel, so

$$\frac{\partial}{\partial t} \mathrm{Var}(x_{t,T}) = 2 (am(t, T) + \gamma(t, T))^2 .$$

Step 4: Substitute the mean branch.

From Statement 3.3, under the baseline exponential equilibrium curve,

$$m(t, T) = x_\infty + Ae^{-\lambda^2(T-t)} + D(t, T),$$

where

$$D(t, T) := \left(x_{t_0,T} - x_\infty - Ae^{-\lambda^2(T-t_0)}\right) e^{-a^2(t-t_0)}.$$

Therefore

$$am(t, T) + \gamma(t, T) = a \left(x_\infty + Ae^{-\lambda^2(T-t)} + D(t, T)\right) - ax_\infty - \left(a + \frac{\lambda^2}{a}\right) Ae^{-\lambda^2(T-t)}.$$

After cancellation,

$$am(t, T) + \gamma(t, T) = aD(t, T) - \frac{\lambda^2}{a} Ae^{-\lambda^2(T-t)}.$$

Thus,

$$\frac{\partial}{\partial t} \mathrm{Var}(x_{t,T}) = 2 \left(aD(t, T) - \frac{\lambda^2}{a} Ae^{-\lambda^2(T-t)}\right)^2 .$$

Step 5: Expanded final form.

Substituting the definition of $D(t, T)$, the variance-rate formula is

$$\boxed{\frac{\partial}{\partial t} \mathrm{Var}(x_{t,T}) = 2 \left[a \left(x_{t_0,T} - x_\infty - Ae^{-\lambda^2(T-t_0)}\right) e^{-a^2(t-t_0)} - \frac{\lambda^2}{a} Ae^{-\lambda^2(T-t)}\right]^2 .} \tag{A.6.1}$$

This proves Statement 3.6.

A.7 Proof of Statement 3.7

Goal. Derive the market signal-to-noise ratio.

Step 1: Start from the variance rate.

From Appendix A.6, the instantaneous variance-rate is

$$\frac{\partial}{\partial t}\operatorname{Var}(x_{t,T}) = 2\left(aD(t,T) - \frac{\lambda^2}{a}Ae^{-\lambda^2(T-t)}\right)^2,$$

where

$$D(t,T) := \left(x_{t_0,T} - x_\infty - Ae^{-\lambda^2(T-t_0)}\right)e^{-a^2(t-t_0)}.$$

Hence the instantaneous noise scale is

$$\sqrt{\frac{\partial}{\partial t}\operatorname{Var}(x_{t,T})} = \sqrt{2}\left|aD(t,T) - \frac{\lambda^2}{a}Ae^{-\lambda^2(T-t)}\right|.$$

Step 2: Evaluate the drift on the mean branch.

From Appendix A.5, the interim drift is

$$f(x,t,T) = a^2x - a^2x_\infty - (\lambda^2 + a^2)Ae^{-\lambda^2(T-t)}.$$

The mean branch is

$$m(t,T) = x_\infty + Ae^{-\lambda^2(T-t)} + D(t,T).$$

Evaluating the drift along this mean branch gives

$$f(m(t,T),t,T) = a^2\left(x_\infty + Ae^{-\lambda^2(T-t)} + D(t,T)\right) - a^2x_\infty - (\lambda^2 + a^2)Ae^{-\lambda^2(T-t)}.$$

After cancellation,

$$f(m(t,T),t,T) = a^2D(t,T) - \lambda^2Ae^{-\lambda^2(T-t)}.$$

Equivalently,

$$f(m(t,T),t,T) = a\left(aD(t,T) - \frac{\lambda^2}{a}Ae^{-\lambda^2(T-t)}\right).$$

Therefore the instantaneous signal scale, defined as the magnitude of the drift along the mean branch, is

$$|f(m(t,T),t,T)| = a\left|aD(t,T) - \frac{\lambda^2}{a}Ae^{-\lambda^2(T-t)}\right|.$$

Step 3: Form the ratio.

The Market Signal-to-Noise Ratio is therefore

$$\mathrm{MSNR} = \frac{|f(m(t,T),t,T)|}{\sqrt{\frac{\partial}{\partial t}\operatorname{Var}(x_{t,T})}} = \frac{a\left|aD(t,T) - \frac{\lambda^2}{a}Ae^{-\lambda^2(T-t)}\right|}{\sqrt{2}\left|aD(t,T) - \frac{\lambda^2}{a}Ae^{-\lambda^2(T-t)}\right|}.$$

Thus,

$$\boxed{\mathrm{MSNR} = \frac{a}{\sqrt{2}}.}$$

Step 4: Expanded form.

Equivalently, substituting the definition of $D(t,T)$, the instantaneous signal scale is

$$|f(m(t,T),t,T)| = a\left|a\left(x_{t_0,T} - x_\infty - Ae^{-\lambda^2(T-t_0)}\right)e^{-a^2(t-t_0)} - \frac{\lambda^2}{a}Ae^{-\lambda^2(T-t)}\right|,$$

and the instantaneous noise scale is

$$\sqrt{\frac{\partial}{\partial t}\operatorname{Var}(x_{t,T})} = \sqrt{2}\left|a\left(x_{t_0,T} - x_\infty - Ae^{-\lambda^2(T-t_0)}\right)e^{-a^2(t-t_0)} - \frac{\lambda^2}{a}Ae^{-\lambda^2(T-t)}\right|.$$

Their ratio is therefore again

$$\boxed{\mathrm{MSNR} = \frac{a}{\sqrt{2}}.}$$

This proves Statement 3.7.

Appendix B KFE Solution in the Equilibrium Regime

B.1 Full Frobenius Proof of the equilibrium regime PDF

We begin from the transformed KFE in the shifted variable

$$\widetilde{x}_{t,T} = x_{t,T} - x_\infty\left(1 - e^{-\lambda^2(T-t)}\right),$$

so that the equilibrium mean has been removed, but without yet imposing any relation between λ and a. In these coordinates the KFE takes the form

$$\frac{\partial\varphi}{\partial t} = a^2\widetilde{x}_{t,T}^2\varphi_{\widetilde{x}_{t,T}\widetilde{x}_{t,T}} + 5a^2\widetilde{x}_{t,T}\,\varphi_{\widetilde{x}_{t,T}} + 3a^2\varphi - \lambda^2 x_\infty e^{-\lambda^2(T-t)}\varphi_{\widetilde{x}_{t,T}}. \tag{B.1.1}$$

To analyse the singular structure at $\widetilde{x}_{t,T} = 0$, introduce the reciprocal variable

$$z = \frac{1}{\widetilde{x}_{t,T}}, \qquad z_{\widetilde{x}_{t,T}} = -z^2, \qquad z_{\widetilde{x}_{t,T}\widetilde{x}_{t,T}} = 2z^3.$$

Using

$$\varphi_{\widetilde{x}_{t,T}} = -z^2\varphi_z, \qquad \varphi_{\widetilde{x}_{t,T}\widetilde{x}_{t,T}} = z^4\varphi_{zz} + 2z^3\varphi_z,$$

equation (B.1.1) becomes

$$\frac{\partial\varphi}{\partial t} = a^2z^2\varphi_{zz} - 3a^2z\varphi_z + 3a^2\varphi + \lambda^2 x_\infty e^{-\lambda^2(T-t)}z^2\varphi_z. \tag{B.1.2}$$

Step 1: Frobenius expansion.

Seek a Frobenius series of the form

$$\varphi(z,t,T) = \sum_{i=0}^{\infty} c_i(t)\,z^{i+m}.$$

Then

$$\varphi_z = \sum_{i=0}^{\infty}(i+m)c_i(t)\, z^{i+m-1}, \qquad \varphi_{zz} = \sum_{i=0}^{\infty}(i+m)(i+m-1)c_i(t)\, z^{i+m-2}.$$

Substituting into (B.1.2) and matching coefficients of z^{i+m} yields

$$\frac{\partial c_i}{\partial t} = a^2(i+m-1)(i+m-3)c_i + \lambda^2 x_\infty e^{-\lambda^2(T-t)}(i+m-1)c_{i-1}, \tag{B.1.3}$$

with $c_{-1} \equiv 0$.

Step 2: Separate the time dependence.

Write

$$c_i(t) = c_i^*\, x_\infty^{i+n} e^{-(i+n)\lambda^2(T-t)}.$$

Then

$$\frac{\partial c_i}{\partial t} = (i+n)\lambda^2 c_i(t).$$

Substituting into (B.1.3) and cancelling the common factor $x_\infty^{i+n}e^{-(i+n)\lambda^2(T-t)}$ gives

$$(i+n)\lambda^2 c_i^* = a^2(i+m-1)(i+m-3)c_i^* + \lambda^2(i+m-1)c_{i-1}^*.$$

At $i=0$, since $c_{-1}^* = 0$, this reduces to the indicial relation

$$n\,\frac{\lambda^2}{a^2} = (m-1)(m-3). \tag{B.1.4}$$

For $i \geq 1$, subtracting (B.1.4) gives

$$\lambda^2 i\, c_i^* = a^2 i(i+2m-4)c_i^* + \lambda^2(i+m-1)c_{i-1}^*,$$

hence

$$\frac{c_i^*}{c_{i-1}^*} = -\frac{\frac{\lambda^2}{a^2}(i+m-1)}{i\left(i+2m-4-\frac{\lambda^2}{a^2}\right)}. \tag{B.1.5}$$

Step 3: Characterise the non-terminating admissible branch.

Let

$$r := \frac{\lambda^2}{a^2}.$$

For the Frobenius series to sum to a single inverse-gamma-type branch, the coefficient ratio needs to have the form

$$\frac{c_i^*}{c_{i-1}^*} = -\frac{\beta}{i}$$

for some constant $\beta > 0$; equivalently, the additional affine factor in (B.1.5) needs to reduce to an i-independent constant. Since the numerator and denominator in (B.1.5) are both affine in i with the same slope, this occurs if and only if they are identical, namely

$$i + m - 1 = i + 2m - 4 - r.$$

Therefore

$$m = 3 + r. \tag{B.1.6}$$

Substituting (B.1.6) into (B.1.5) gives

$$\frac{c_i^*}{c_{i-1}^*} = -\frac{r}{i}, \qquad c_i^* = \frac{(-r)^i}{i!}\, c_0^*. \tag{B.1.7}$$

The indicial relation (B.1.4) then yields

$$n = r + 2. \tag{B.1.8}$$

Hence the admissible branch is the one-parameter family

$$\varphi(z,t,T) = c_0^* x_\infty^{r+2} z^{r+3} e^{-(r+2)\lambda^2(T-t)} \exp\bigl[-r x_\infty e^{-\lambda^2(T-t)} z\bigr].$$

Returning to $\widetilde{x}_{t,T} = 1/z$, this becomes

$$\varphi(\widetilde{x}_{t,T},t,T) = c_0^* x_\infty^{r+2} e^{-(r+2)\lambda^2(T-t)} \widetilde{x}_{t,T}^{-(r+3)} \exp\left(-\frac{r x_\infty e^{-\lambda^2(T-t)}}{\widetilde{x}_{t,T}}\right). \tag{B.1.9}$$

Thus the Frobenius construction yields an inverse-gamma family with shape parameter

$$\alpha = r + 2.$$

Normalisation of the general branch.

Equation (B.1.9) already identifies the admissible inverse-gamma family. Since $\lambda^2 = ra^2$, it may be written as

$$\varphi(\widetilde{x}_{t,T},t,T) = c_0^* \frac{x_\infty^{r+2} e^{-(r+2)ra^2(T-t)}}{\widetilde{x}_{t,T}^{r+3}} \exp\left(-\frac{r x_\infty e^{-ra^2(T-t)}}{\widetilde{x}_{t,T}}\right).$$

Normalisation fixes

$$c_0^* = \frac{r^{r+2}}{\Gamma(r+2)}.$$

Therefore the maturity-indexed equilibrium density is

$$\boxed{\varphi(\widetilde{x}_{t,T}^{(r)},t,T) = \frac{\left(r x_\infty e^{-ra^2(T-t)}\right)^{r+2}}{\Gamma(r+2)\left(\widetilde{x}_{t,T}^{(r)}\right)^{r+3}} \exp\left(-\frac{r x_\infty e^{-ra^2(T-t)}}{\widetilde{x}_{t,T}^{(r)}}\right), \qquad \widetilde{x}_{t,T}^{(r)} > 0.} \tag{B.1.10}$$

Equivalently,

$$\widetilde{x}_{t,T}^{(r)} \sim \mathrm{InvGamma}\Bigl(r+2,\ r x_\infty e^{-ra^2(T-t)}\Bigr).$$

The order-three density used in the one-scale case is recovered by setting $r = 1$, so that $\lambda^2 = a^2$. In that specialization,

$$\boxed{\varphi(\widetilde{x}_{t,T},t,T) = \frac{x_\infty^3 e^{-3a^2(T-t)}}{\Gamma(3)\,\widetilde{x}_{t,T}^4} \exp\left(-\frac{x_\infty e^{-a^2(T-t)}}{\widetilde{x}_{t,T}}\right)} \tag{B.1.12}$$

which is the shifted inverse-gamma distribution of order 3.

B.2 Direct Laplace-Transform Derivation of the General r-Branch Laplace Transform

Let

$$\alpha := r + 2, \qquad \beta^{(r)}_{t,T} := r x_\infty e^{-ra^2(T-t)}.$$

From (B.1.10), the shifted variable $\widetilde{x}^{(r)}_{t,T}$ has density

$$\varphi(\widetilde{x}^{(r)}_{t,T}, t, T) = \frac{\left(\beta^{(r)}_{t,T}\right)^{\alpha}}{\Gamma(\alpha)\left(\widetilde{x}^{(r)}_{t,T}\right)^{\alpha+1}} \exp\left(-\frac{\beta^{(r)}_{t,T}}{\widetilde{x}^{(r)}_{t,T}}\right), \qquad \widetilde{x}^{(r)}_{t,T} > 0.$$

We compute its Laplace transform directly. For $q > 0$, define

$$\widehat{\varphi}_r(q, t, T) = \int_0^\infty e^{-q\widetilde{x}^{(r)}_{t,T}} \varphi(\widetilde{x}^{(r)}_{t,T}, t, T)\, d\widetilde{x}^{(r)}_{t,T}.$$

Substituting the density gives

$$\widehat{\varphi}_r(q, t, T) = \frac{\left(\beta^{(r)}_{t,T}\right)^{\alpha}}{\Gamma(\alpha)} \int_0^\infty \left(\widetilde{x}^{(r)}_{t,T}\right)^{-\alpha-1} \exp\left(-q\widetilde{x}^{(r)}_{t,T} - \frac{\beta^{(r)}_{t,T}}{\widetilde{x}^{(r)}_{t,T}}\right) d\widetilde{x}^{(r)}_{t,T}. \tag{B.2.1}$$

Using the standard integral identity

$$\int_0^\infty x^{\nu-1} \exp(-\mu x - \eta/x)\ dx = 2\left(\frac{\eta}{\mu}\right)^{\nu/2} K_\nu(2\sqrt{\mu\eta}),$$

with $\nu = -\alpha$, $\mu = q$, and $\eta = \beta^{(r)}_{t,T}$, and using $K_{-\nu} = K_\nu$, we obtain

$$\widehat{\varphi}_r(q, t, T) = \frac{2}{\Gamma(\alpha)} \left(\beta^{(r)}_{t,T} q\right)^{\alpha/2} K_\alpha\left(2\sqrt{\beta^{(r)}_{t,T} q}\right). \tag{B.2.2}$$

Substituting $\alpha = r + 2$ and $\beta^{(r)}_{t,T} = r x_\infty e^{-ra^2(T-t)}$ gives

$$\boxed{\widehat{\varphi}_r(q, t, T) = \frac{2}{\Gamma(r+2)} \left(r x_\infty e^{-ra^2(T-t)} q\right)^{(r+2)/2} K_{r+2}\left(2\sqrt{r x_\infty e^{-ra^2(T-t)} q}\right), \qquad q > 0.} \tag{B.2.3}$$

If one prefers the moment-generating-function variable s, write $q = -s$. Then the transform is well defined for $s < 0$ and becomes

$$\boxed{\widetilde{\varphi}_r(s, t, T) = \frac{2}{\Gamma(r+2)} \left(-r x_\infty e^{-ra^2(T-t)} s\right)^{(r+2)/2} K_{r+2}\left(2\sqrt{-r x_\infty e^{-ra^2(T-t)} s}\right), \qquad s < 0.} \tag{B.2.4}$$

The order-three expression is recovered by setting $r = 1$. Since $\Gamma(3) = 2$, (B.2.3) then reduces to

$$\widehat{\varphi}_1(q, t, T) = \left(x_\infty e^{-a^2(T-t)} q\right)^{3/2} K_3\left(2\sqrt{x_\infty e^{-a^2(T-t)} q}\right).$$

Thus the previous order-three transform is the one-scale specialization of the general r-branch Laplace transform.

B.3 Derivation of the General r-Branch Expectation Formula

Goal. Derive the closed-form expression for the conditional expectation $\mathbb{E}[x_{t,T}]$ under the general r-branch, using only the first-moment identity implied by the Kolmogorov Forward Equation.

Context. Appendix A.3 establishes the general expectation dynamics under affine drift. In this appendix we derive the explicit closed-form expectation after imposing the baseline drift–diffusion identification $a_1 = a^2$, while retaining the general maturity ratio

$$r = \frac{\lambda^2}{a^2} > 0.$$

Step 1: First-moment identity. For fixed maturity T, the Kolmogorov Forward Equation implies

$$\frac{\partial}{\partial t}\mathbb{E}[x_{t,T}] = -\mathbb{E}[f(x_{t,T}, t, T)]\,. \tag{B.3.1}$$

Step 2: General r-branch drift. Under the general r-branch drift–diffusion specification in Section 5.3, the drift is

$$f(x_{t,T}, t, T) = a^2 x_{t,T} - a^2 x_\infty \left(1 - e^{-ra^2(T-t)}\right). \tag{B.3.2}$$

Step 3: Taking expectations. Taking expectations of (B.3.2) and using linearity yields

$$\mathbb{E}[f(x_{t,T}, t, T)] = a^2\mathbb{E}[x_{t,T}] - a^2 x_\infty \left(1 - e^{-ra^2(T-t)}\right).$$

Substituting into (B.3.1) gives the evolution equation

$$\frac{\partial}{\partial t}\mathbb{E}[x_{t,T}] + a^2\mathbb{E}[x_{t,T}] = a^2 x_\infty - a^2 x_\infty e^{-ra^2(T-t)}. \tag{B.3.3}$$

Step 4: Integrating factor. Equation (B.3.3) is a first-order linear ODE in $\mathbb{E}[x_{t,T}]$. The integrating factor is $e^{a^2 t}$, and multiplying (B.3.3) by this factor yields

$$\frac{\partial}{\partial t}\left(e^{a^2 t}\mathbb{E}[x_{t,T}]\right) = a^2 x_\infty e^{a^2 t} - a^2 x_\infty e^{-ra^2 T} e^{(1+r)a^2 t}.$$

Step 5: Integration in time. Integrating from t_0 to t gives

$$\begin{aligned} e^{a^2 t}\mathbb{E}[x_{t,T}] - e^{a^2 t_0}\mathbb{E}[x_{t_0,T}] &= x_\infty\left(e^{a^2 t} - e^{a^2 t_0}\right) \\ &\quad - \frac{x_\infty}{1+r} e^{-ra^2 T}\left(e^{(1+r)a^2 t} - e^{(1+r)a^2 t_0}\right). \end{aligned}$$

Step 6: Solving for the expectation. Dividing through by $e^{a^2 t}$ yields

$$\begin{aligned} \mathbb{E}[x_{t,T}] &= \mathbb{E}[x_{t_0,T}]e^{-a^2(t-t_0)} + x_\infty\left(1 - e^{-a^2(t-t_0)}\right) \\ &\quad - \frac{x_\infty}{1+r} e^{-ra^2(T-t)} + \frac{x_\infty}{1+r} e^{-a^2(t-t_0)} e^{-ra^2(T-t_0)}. \end{aligned} \tag{B.3.4}$$

Step 7: Imposing the initial condition. Using the general r-branch forward-curve representation at time t_0,

$$\mathbb{E}[x_{t_0,T}] = x_{t_0,T} = x_\infty - \frac{x_\infty}{1+r} e^{-ra^2(T-t_0)} + \left(x_{t_0,t_0} - \frac{r}{1+r} x_\infty \right) e^{-a^2(T-t_0)},$$

and simplifying (B.3.4), we obtain

$$\boxed{\mathbb{E}[x_{t,T}] = x_\infty - \frac{x_\infty}{1+r} e^{-ra^2(T-t)} + \left(x_{t_0,t_0} - \frac{r}{1+r} x_\infty \right) e^{-a^2(T+t-2t_0)}.} \tag{B.3.5}$$

Equivalently, writing

$$C_r := x_{t_0,t_0} - \frac{r}{1+r} x_\infty,$$

the expectation may be expressed compactly as

$$\boxed{\mathbb{E}[x_{t,T}] = x_\infty - \frac{x_\infty}{1+r} e^{-ra^2(T-t)} + C_r e^{-a^2(T+t-2t_0)}.} \tag{B.3.6}$$

For $r = 1$, this reduces to the one-scale expression associated with the order-three branch.

Appendix C Relation to Conventional Interest Rates

This appendix maps the instantaneous rolling forward rate (IRFR) to the conventional continuously compounded rate $r_{t,T}$ over the remaining maturity interval $[t, T]$. Although the conventional rate is indexed by the fixed maturity date T, the IRFRs entering the maturity average continue to roll with calendar time. The purpose of this appendix is to transfer the general r-branch expectation and local covariance-rate results for the IRFR to the corresponding conventional rate.

Throughout this appendix, write

$$\tau := T - t, \qquad \Delta := t - t_0, \qquad k := a^2, \qquad r := \frac{\lambda^2}{a^2} > 0,$$

and define the short-end displacement

$$C_r := x_{t_0,t_0} - \frac{r}{1+r} x_\infty. \tag{C.0.1}$$

C.1 Conventional Rate as a Maturity Average

On a discrete tenor grid $t = S_0 < S_1 < \cdots < S_N = T$ with mesh ΔS and $N\Delta S = T - t$, the conventional continuously compounded rate is approximated by

$$r_{t,T} = \frac{1}{T-t} \sum_{j=1}^{N} x_{t,S_j} \, \Delta S.$$

In continuous time,

$$r_{t,T} = \frac{1}{T-t}\int_t^T x_{t,S}\, dS. \tag{C.1.1}$$

Using the general r-branch forward-curve representation (5.5.1), the IRFR curve at time t may be written, for $S \geq t$, as

$$x_{t,S} = x_\infty - \frac{x_\infty}{1+r}e^{-rk(S-t)} + \left(x_{t,t} - \frac{r}{1+r}x_\infty\right)e^{-k(S-t)}. \tag{C.1.2}$$

Therefore the conventional rate itself is

$$r_{t,T} = x_\infty - \frac{x_\infty}{1+r}\frac{1-e^{-rk\tau}}{rk\tau} + \left(x_{t,t} - \frac{r}{1+r}x_\infty\right)\frac{1-e^{-k\tau}}{k\tau}. \tag{C.1.3}$$

Thus the conventional rate inherits the two-exponential maturity structure of the general r-branch. When $r = 1$, the two maturity loadings collapse to the same exponential scale.

C.2 Expected Value of the Conventional Rate

From the general r-branch expectation formula in Section 5.6, for any maturity date $S \geq t$,

$$E[x_{t,S}] = x_\infty - \frac{x_\infty}{1+r}e^{-rk(S-t)} + C_r e^{-k(S+t-2t_0)}. \tag{C.2.1}$$

Equivalently, since $S + t - 2t_0 = (S-t) + 2(t-t_0)$,

$$E[x_{t,S}] = x_\infty - \frac{x_\infty}{1+r}e^{-rk(S-t)} + C_r e^{-2k\Delta}e^{-k(S-t)}. \tag{C.2.2}$$

Integrating over maturities gives

$$\begin{aligned} E\left[\int_t^T x_{t,S}\, dS\right] &= \int_t^T \left[x_\infty - \frac{x_\infty}{1+r}e^{-rk(S-t)} + C_r e^{-2k\Delta}e^{-k(S-t)}\right] dS \\ &= x_\infty\tau - \frac{x_\infty}{1+r}\frac{1-e^{-rk\tau}}{rk} + C_r e^{-2k\Delta}\frac{1-e^{-k\tau}}{k}. \end{aligned} \tag{C.2.3}$$

Using (C.1.1), we obtain

$$E[r_{t,T}] = x_\infty - \frac{x_\infty}{1+r}\frac{1-e^{-rk\tau}}{rk\tau} + C_r e^{-2k\Delta}\frac{1-e^{-k\tau}}{k\tau}. \tag{C.2.4}$$

Substituting back $C_r = x_{t_0,t_0} - rx_\infty/(1+r)$ gives

$$E[r_{t,T}] = x_\infty - \frac{x_\infty}{1+r}\frac{1-e^{-ra^2(T-t)}}{ra^2(T-t)} + \left(x_{t_0,t_0} - \frac{r}{1+r}x_\infty\right)e^{-2a^2(t-t_0)}\frac{1-e^{-a^2(T-t)}}{a^2(T-t)}. \tag{C.2.5}$$

For $r = 1$, this reduces to the one-scale expression obtained from the order-three branch.

C.3 Instantaneous Variance of the Conventional Rate

We next compute the local stochastic variance rate of the conventional rate using the covariance-rate kernel induced by the one-factor IRFR dynamics. From the general r-branch IRFR variance-rate formula, define

$$H(S) := C_r e^{-k(S+t-2t_0)} + \frac{r}{1+r} x_\infty e^{-rk(S-t)}. \tag{C.3.1}$$

Then

$$\frac{\partial}{\partial t} \mathrm{Var}(x_{t,S}) = 2a^2 H(S)^2. \tag{C.3.2}$$

Because the IRFR dynamics are one-factor, the instantaneous covariance-rate kernel between two forward maturities S_1 and S_2 is

$$\frac{\partial}{\partial t} \mathrm{Cov}(x_{t,S_1}, x_{t,S_2}) = 2a^2 H(S_1) H(S_2). \tag{C.3.3}$$

Consequently,

$$\begin{aligned} \frac{\partial}{\partial t} \mathrm{Var}\left(\int_t^T x_{t,S}\, dS \right) &= \int_t^T \int_t^T \frac{\partial}{\partial t} \mathrm{Cov}(x_{t,S_1}, x_{t,S_2})\, dS_1\, dS_2 \\ &= 2a^2 \left(\int_t^T H(S)\, dS \right)^2. \end{aligned} \tag{C.3.4}$$

Using $S + t - 2t_0 = (S - t) + 2\Delta$, the integral of H is

$$\begin{aligned} \int_t^T H(S)\, dS &= C_r e^{-2k\Delta} \int_t^T e^{-k(S-t)}\, dS + \frac{r}{1+r} x_\infty \int_t^T e^{-rk(S-t)}\, dS \\ &= C_r e^{-2k\Delta} \frac{1 - e^{-k\tau}}{k} + \frac{x_\infty}{1+r} \frac{1 - e^{-rk\tau}}{k}. \end{aligned} \tag{C.3.5}$$

Finally, using (C.1.1), the instantaneous variance-rate of the conventional rate is

$$\frac{\partial}{\partial t} \mathrm{Var}(r_{t,T}) = \frac{2}{a^2 (T-t)^2} \left[C_r e^{-2a^2(t-t_0)} \left(1 - e^{-a^2(T-t)} \right) + \frac{x_\infty}{1+r} \left(1 - e^{-ra^2(T-t)} \right) \right]^2. \tag{C.3.6}$$

Equivalently, substituting back the definition of C_r,

$$\begin{aligned} \frac{\partial}{\partial t} \mathrm{Var}(r_{t,T}) = \frac{2}{a^2(T-t)^2} \Bigg[&\left(x_{t_0,t_0} - \frac{r}{1+r} x_\infty \right) e^{-2a^2(t-t_0)} \left(1 - e^{-a^2(T-t)} \right) \\ &+ \frac{x_\infty}{1+r} \left(1 - e^{-ra^2(T-t)} \right) \Bigg]^2. \end{aligned} \tag{C.3.7}$$

For $r = 1$, the two exponential terms share the same maturity loading, and (C.3.7) collapses to the one-scale expression associated with the order-three branch. For $r \neq 1$, the conventional rate inherits the two-scale maturity structure of the IRFR curve while preserving a one-factor covariance kernel.

Appendix D No-Arbitrage Motivation for Rolling Maturity Invariance

D.1 Objective

This appendix gives the no-arbitrage motivation for Axiom 2 (Rolling–Maturity Invariance). The claim is intentionally stated as a consistency condition rather than as a complete replication theorem. If two positions represent the same remaining-maturity exposure, then their local law should not depend on the calendar date at which that exposure is implemented, except through explicitly specified changes in the underlying bond-price dynamics or market price of risk. Axiom 2 imposes this rolling consistency directly by requiring the IRFR dynamics to depend on $(x, T-t)$, rather than separately on (x, t, T).

D.2 IRFR and Bond-Price Consistency

The IRFR is linked to zero-coupon bond prices by

$$\int_t^T x_{t,S}\, dS = -\log P(t,T).$$

Thus the curve $S \mapsto x_{t,S}$ determines the bond price $P(t,T)$, and local changes in the curve correspond to local changes in the family of bond prices. In this sense, IRFR dynamics are not merely auxiliary modelling objects: they must be consistent with the P&L generated by rolling and holding zero-coupon bond exposures.

D.3 Hold and Roll Implementations

Fix a remaining maturity $\tau > 0$ and set $T = t + \tau$. Consider two local implementations of a forward exposure over a short interval $[t, t+\Delta t]$.

Hold implementation. Enter exposure to $x_{t,T}$ and hold it over $[t, t + \Delta t]$. The local change is represented by

$$\Delta V^{\mathrm{hold}} = x_{t+\Delta t,T} - x_{t,T}.$$

Roll implementation. Maintain exposure to the same remaining maturity τ by rolling the maturity date from T to $T + \Delta t$. The local change is represented by

$$\Delta V^{\mathrm{roll}} = x_{t+\Delta t,T+\Delta t} - x_{t,T}.$$

These two expressions need not be pathwise identical, because they refer to slightly different maturity dates after the roll. The no-arbitrage requirement is weaker and more precise: absent a compensating change in the traded bond dynamics or in risk premia, the two implementations of the same remaining-maturity exposure must have the same local stochastic law.

D.4 Rolling Consistency Requirement

Rolling consistency therefore requires

$$\mathcal{L}\big(\Delta V^{\text{hold}} \mid x_{t,T} = x,\, T - t = \tau\big) = \mathcal{L}\big(\Delta V^{\text{roll}} \mid x_{t,T} = x,\, T - t = \tau\big) + o(\Delta t),$$

where $\mathcal{L}$ denotes the local conditional law. In diffusion form, this means that the local drift and volatility of an exposure with state x and remaining maturity τ must be invariant under the simultaneous shift

$$(t, T) \mapsto (t + \Delta t, T + \Delta t).$$

Equivalently, the coefficients may depend on calendar time and maturity only through their difference:

$$\mu(x, t, T) = \mu(x, T - t), \qquad \sigma(x, t, T) = \sigma(x, T - t).$$

This is precisely Axiom 2.

D.5 What Goes Wrong Without Invariance

If, instead, two contracts with the same state x and the same remaining maturity τ have different local coefficients at different calendar dates, then rolling introduces an implementation-dependent local law:

$$\mu(x, t, T) \neq \mu(x, t + \Delta t, T + \Delta t) \quad \text{or} \quad \sigma(x, t, T) \neq \sigma(x, t + \Delta t, T + \Delta t).$$

In that case, the model assigns different instantaneous expected P&L or risk to what is economically the same remaining-maturity exposure. Such a discrepancy is a rolling inconsistency. In a complete bond-market specification it would have to be offset by an explicit adjustment elsewhere–for example, in the bond-price law, the financing convention, or the market price of risk. Since the present framework does not introduce such compensating terms, Axiom 2 rules out the inconsistency at the level of the IRFR coefficients.

D.6 Interpretation

Axiom 2 can therefore be read as a path-independence condition:

> The local law of a forward-rate exposure should be determined by its state and remaining maturity, not by the calendar date on which the exposure is implemented.

This prevents the model from assigning different dynamics to equivalent rolling implementations of the same maturity exposure.

D.7 Conclusion

Rolling–maturity invariance is therefore a structural no-arbitrage consistency condition. It does not assert, by itself, a fully specified arbitrage portfolio in every enlarged market model. Rather, it states that in the absence of additional compensating risk-premium or bond-dynamics terms, the admissible IRFR diffusion coefficients must satisfy

$$\mu(x, t, T) = \mu(x, T - t), \qquad \sigma(x, t, T) = \sigma(x, T - t).$$

This is the content of Axiom 2 and the invariance structure used throughout the paper.

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