IRFR: A One-Factor Interest Rate Model

Motivated by risk, expectations, and rolling maturity

A personal route into the IRFR framework: from duration and parallel shifts, to volatility term structures, to the question of what rate object should be random.

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https://papers.ssrn.com/sol3/papers.cfm?abstract_id=6748347

https://sachinoza56-lgtm.github.io/IRFR-Paper/

Layout of the presentation

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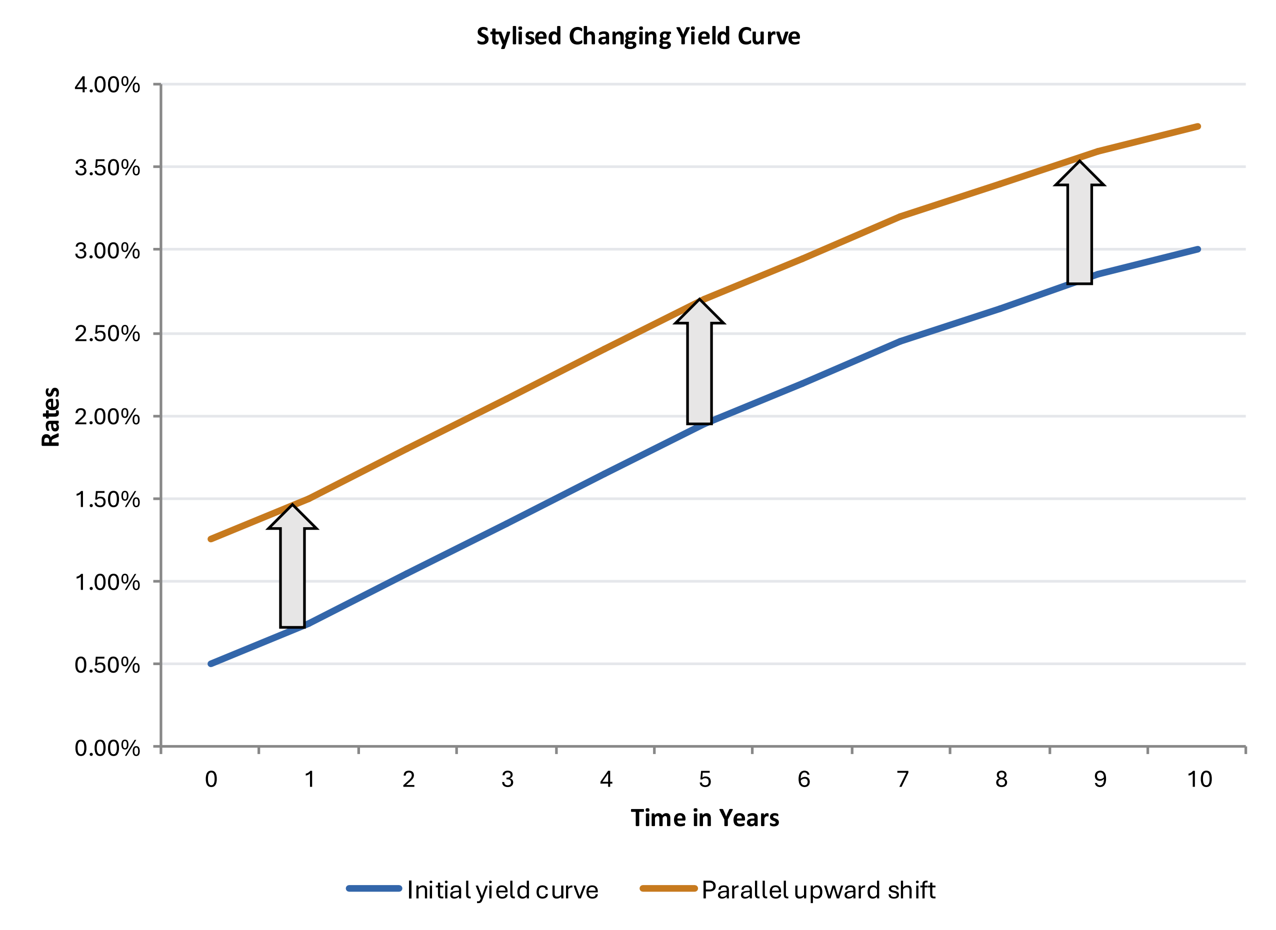
Duration: a tractable first language for interest-rate risk

Parallel shifts are useful, but they compress the whole curve into one movement.

- Duration turns curve risk into a single number.
- That is powerful for first-order risk analysis.
- But it effectively treats the curve as one functional object shifting in parallel.
- Implicit in this shift is that all rates are governed by a single (albeit simple) stochastic process
- Appealing — and incomplete.



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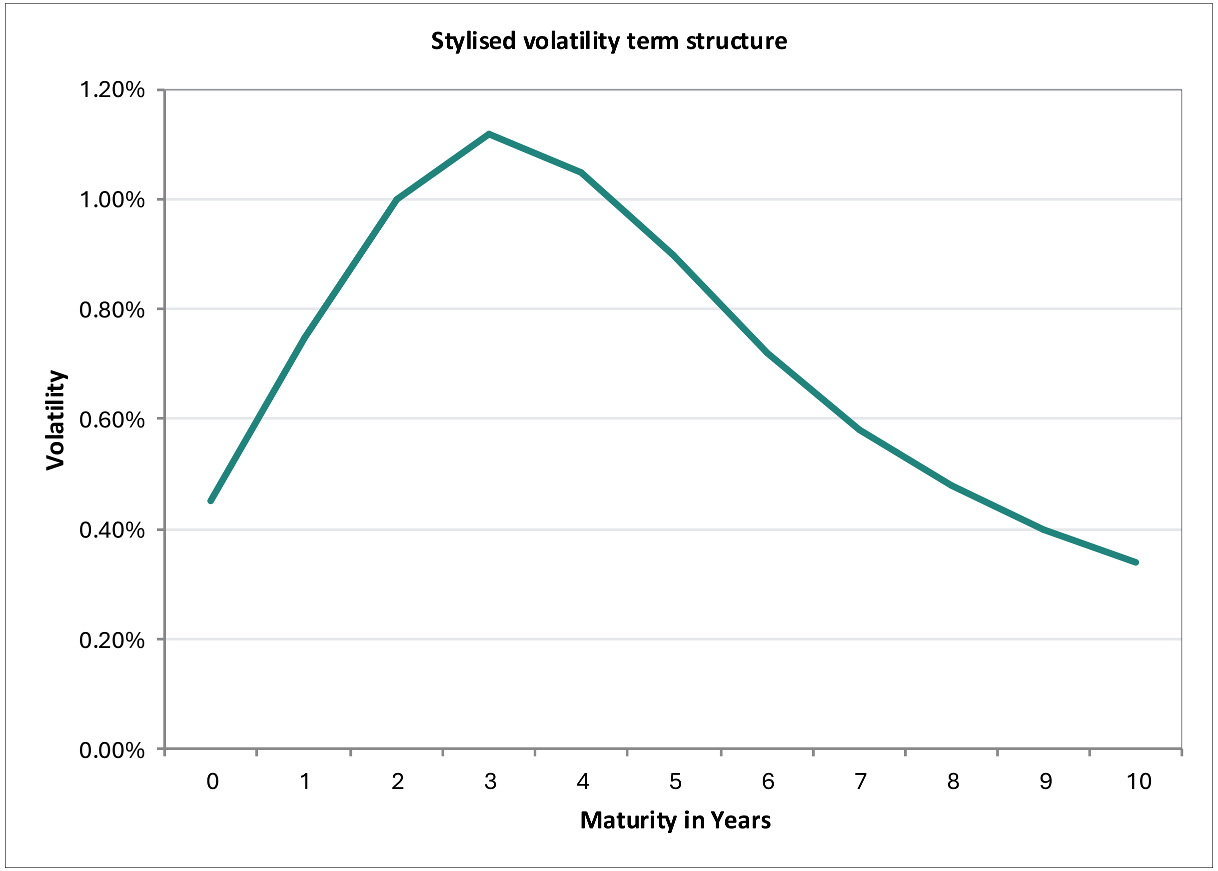
Stylised curves for intuition, not calibration.

Volatility Term Structure: More Complex Relationship

Different maturities do not usually carry the same local uncertainty.

- Interest-rate volatility is often maturity-dependent.
- It may peak away from the very short end.
- That suggests more structure than a simple parallel-shift picture.
- The puzzle: can this happen naturally in a one-factor model?
    - Only possible for non-stationary states



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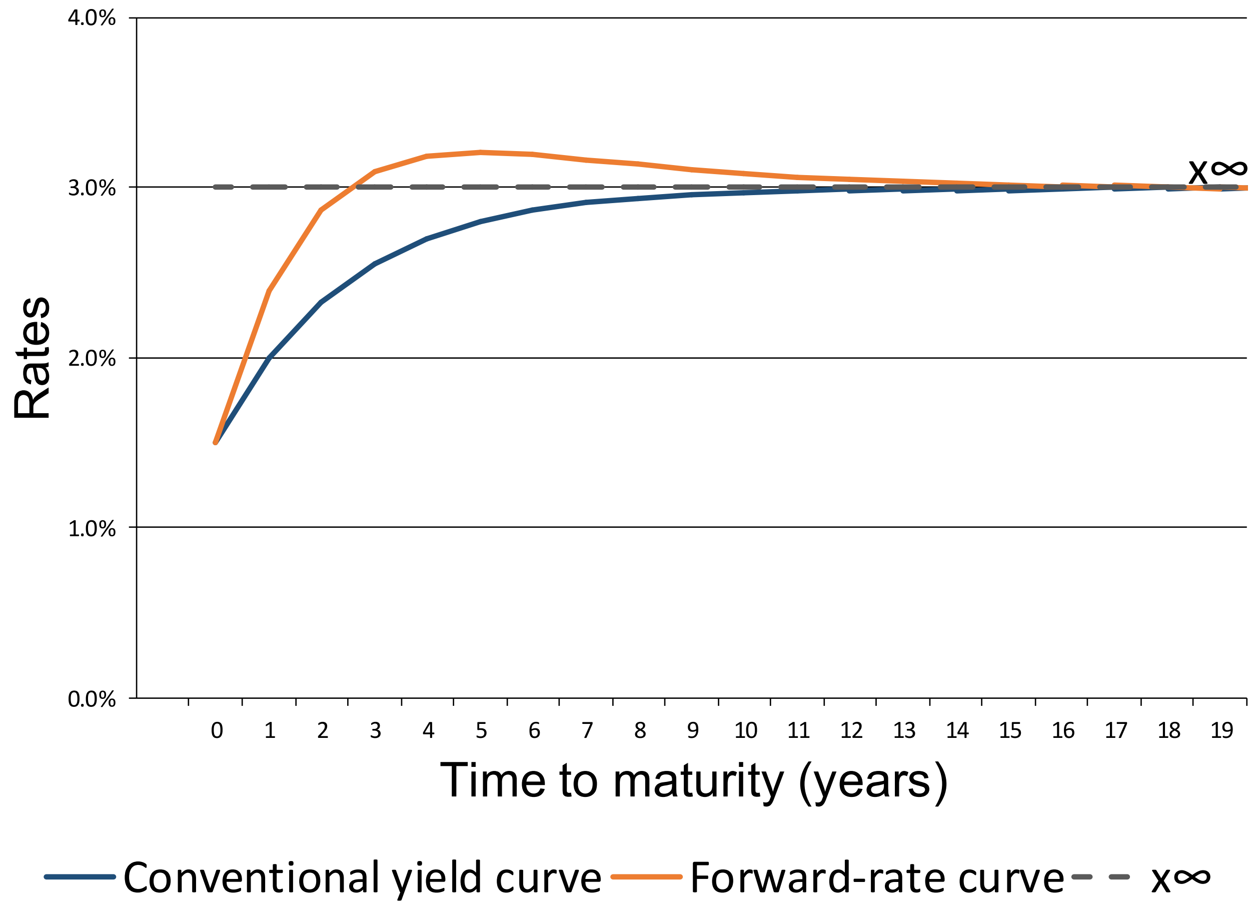
The humped shape is a motif returned to later in the presentation.

Expectations theory: appealing, but incomplete

The curve seems to say something about future rates, but risk cannot be ignored.

- Classical expectations theory says forward rates contain information about future short rates.
- That intuition is appealing.
- But risk premia, liquidity, policy and institutional effects also matter.
- The aim is to preserve the intuition without pretending risk is absent.
- Appealing — and incomplete.



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Both curves are stylised. The point is the distinction between averaged yields and forwards.

The question that follows

Can one framework connect curve shape, risk and the ‘interest rate’ object we choose to model?

Duration
tractable risk, but parallel shifts rationale incomplete

Volatility term structure: maturity-specific risk

Expectations intuition: curve shape should matter

That question motivates another question:
What object should we try to model?

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What object to use?

The modelling primitive should have financial meaning, not just mathematical convenience.

In Black-Scholes, the stock price is the natural object because its change creates P&L.
For interest rates, the analogous object is less obvious.
Possible objects include:

- short rates
- yields
- forwards
- bond prices
- rolling forwards

The choice is not cosmetic - it determines the financial meaning of the model.

The instantaneous rolling forward rate **(IRFR)** is chosen for financial meaning, not novelty.

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For Stock Prices

stock-price change creates trading P&L

For interest rates:

the rolling forward exposure is the object that creates a local P&L

primitive choice -> meaning of dx -> meaning of risk and return

The rolling rate object

Why the primitive should roll, not just move

Illustration

- A buyer and seller enter a unit-notional infinitesimal forward-rate contract linked to a future maturity date T.
- At inception, the contract is struck at fair value, so its initial value is zero.
- Over a small calendar-time interval Δt, the relevant forward-rate object changes by Δx.
- For unit notional, the local change in value is approximately driven by that rate change.

For unit notional, the local change in value is approximately driven by that rate change.

$$\Delta V \approx \Delta x$$

value change ≈ rate-object change

Therefore, x is the natural candidate for the object of interest.

Its infinitesimal change has direct financial meaning: it is the local P&L driver of the infinitesimal rolling forward-rate exposure.

Unlike a static curve label, this primitive rolls in both calendar time and remaining maturity.

We call this primitive the instantaneous rolling forward rate, or IRFR.

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This is the motivation for using a rolling object.

A key observation: the IRFR change relates to P&L

The illustration now becomes a precise rolling-maturity notation.

Object and local change

$$x_{t,T}$$

t = calendar time

T = maturity date

$\tau = T - t$ = remaining maturity

$$\Delta x_{t,T} = x_{t+\Delta t,T} - x_{t,T}$$

For a unit-notional infinitesimal exposure, this local change is the P&L driver.

The roll is not a technical complication. It is the financial object.

The roll

$$\tau = T - t$$

as calendar time advances, remaining maturity shortens

today tomorrow

$$T - t \rightarrow T - (t + \Delta t)$$

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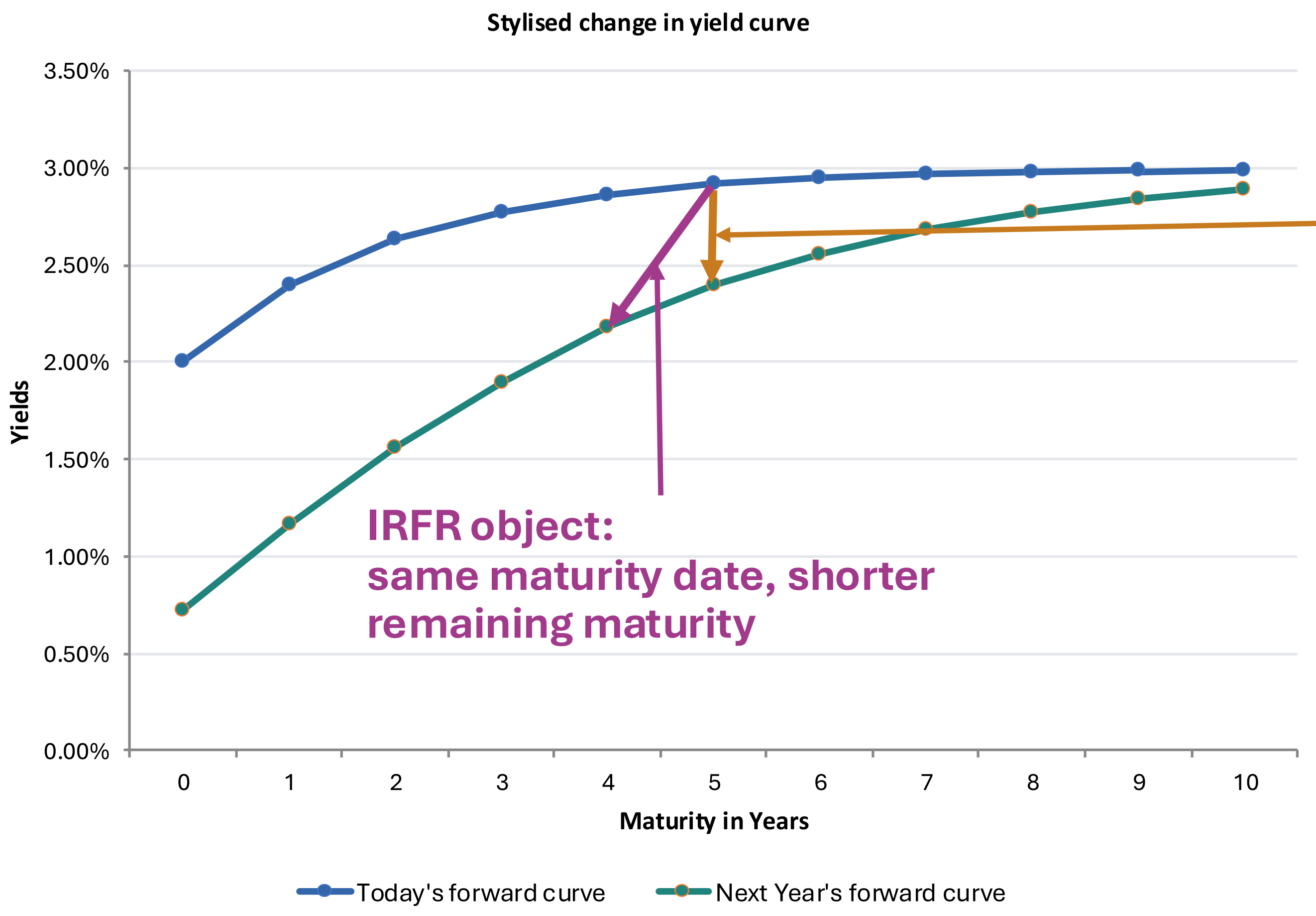
The IRFR behaves like a financial exposure rather than a static curve label.

HJM forwards and the IRFR: same starting point, different roll

One graph, two ways of following a forward-rate exposure from today to next year.



HJM object:
fixed maturity date on next year's curve

Both arrows start from the same forward rate today. The IRFR follows the roll in remaining maturity.

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Stylised diagram: HJM emphasizes the forward curve; the IRFR emphasizes the rolling exposure.

Why write the model via the Kolmogorov Forward Equation (KFE)?

The SDE is useful, but the Kolmogorov Forward Equation (KFE) makes the distributional structure visible.

KFE

$$\frac{\partial\, \varphi(x,t,T)}{\partial t} = -\frac{\partial \left(\mu(x,\tau)\varphi(x,t,T)\right)}{\partial x} + \frac{1}{2}\frac{\partial^2 \left(\sigma^2(x,\tau)\varphi(x,t,T)\right)}{\partial x^2}$$

We use a partial differential equation notation called the Kolmogorov Forward Equation, KFE.

The KFE describes the evolution of probability densities, $\varphi(x,t,T)$.

KFE is natural for expectation and variance calculations.

Most importantly, it more naturally lends itself to solve for the equilibrium probability density function (PDF).

The conventional SDE notation describes paths, but is less intuitive with the workflow described here.

More natural for:

→ application of the axiomatic approach
→ density evolution
→ moments
→ equilibrium PDE: shifted inverse-gamma law

The KFE is not just notation.
It is how the equilibrium law is found.

SDE

$$dx = \mu(x,\tau)dt + \sigma(x,\tau)dW$$

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The next section walks through the formal statements of the paper.

What properties MUST hold for the IRFR

Primitive Object Chosen:
IRFR

Mathematical structure chosen:
KFE

How should we manipulate the IRFR in the KFE form?

The next motivation is :
What minimal set of behaviours do we wish the IRFR to have?
These will be our Axioms

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The axioms as modelling discipline

Each axiom answers a different modelling need.

Axiom 1: local stochastic evolution

The IRFR evolves through a KFE. This provides the dynamic input for drift, volatility and rolling maturity.

Axiom 2: rolling maturity invariance.

Two IRFRs with the same state and remaining maturity should have the same local law. Calendar time enters through $\tau = T - t$

Axiom 3: equilibrium moment transport

At equilibrium, the first and second moments are transported along remaining maturity. For the first moment, this is a modified expectations hypothesis. Recall: the local change in the IRFR is the local return/P&L driver of the rolling exposure.

Axiom 4: positivity at zero maturity

The short-maturity IRFR remains positive, and the diffusion loading vanishes at the zero boundary.

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Axiom 1 local stochastic behaviour

$$\partial_t \varphi(x,t,T_1) = -\partial_x[\mu(x,\tau_1)\varphi(x,t,T_1)] + \frac{1}{2}\partial_{xx}[\sigma^2(x,\tau_1)\varphi(x,t,T_1)]$$

Axiom 2 Full curve stochastic behaviour means φ, μ and σ^2 also govern other maturities

$$\partial_t \varphi(y,t,T_2) = -\partial_y[\mu(y,\tau_2)\varphi(y,t,T_2)] + \frac{1}{2}\partial_{yy}[\sigma^2(y,\tau_2)\varphi(y,t,T_2)]$$

Axiom 3 At Equilibrium stationarity means:

$$\partial_t E[x_{t,T_1}] = -\partial_T E[x_{t,T_1}] \quad \textbf{and} \quad \partial_t Var[x_{t,T_1}] = -\partial_T Var[x_{t,T_1}]$$

Axiom 4 Positivity of short rate

$$x_{t,t} > 0$$

The aim is structure: the model is built around the application of the axioms to the KFE which describes the rolling financial object.

Axioms first; heuristic proof instinct second.

Specific comment on equilibrium – Axiom 3

The opening intuition returns, but with probability and risk made explicit.

- At the outset, expectations theory was appealing because the curve seems to say something about future rates.
- The equilibrium axiom keeps that intuition alive, but does not ignore the role of risk.
- At equilibrium, the expected value of the IRFR after a small time step is the current IRFR rolled down the curve by the same reduction in remaining maturity.
    - $E_t\left[x_{t+\delta t,T}\right] \approx x_{t,T-\delta t}$
- Recall: the P&L driver for an infinitesimal forward-rate exposure is the IRFR's local roll and movement. The associated variance will later be shown to arise from the same rolling structure

Expectations intuition

$\boldsymbol{E[x_\tau] carries\ curve - shape\ information}$

$\boldsymbol{\varphi_e(x,\tau)\ adds\ risk\ and\ dispersion}$

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Key message: equilibrium is not a retreat from expectations theory; it is the risk-aware version of it.

Applying the Axioms

Apply axioms to the KFE to determine the drift and diffusion dynamics

Use the dynamics to determine the key quantities

These include expected values, variance and yield curve shape

The mathematical application will be iterative, but the payoff should be a set of axiomatically consistent properties

Proof section: one statement, three layers

The full proof is in the paper; the deck isolates the mathematical mechanism.

Recommended build structure

1 What is being shown

Name the mathematical claim in plain language before showing the algebra.

2 The key equation / identity

Essential formula in a dedicated text box, so we know what to track.

3 The intuition and payoff

Explanation why the statement matters for the IRFR framework and for the later slides.

Proof order

1 Affine (linear) state dependence of drift and diffusion terms
2 Exponential dependence on time components, t and T of drift and diffusion terms
3 Relation between drift and diffusion parameters
4 Allows many key properties including expected values, inst variance and yield curve shape to be evaluated
5 Market Signal to noise Ratio (MSNR) a global constant – independent of state, calendar time and time to maturity
6 PDF solution to the KFE
7 Role of $\lambda^2 and\ a^2$

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The drift and diffusion terms are linear (affine) with respect to the IRFR

Rolling-maturity invariance plus affine closure restricts state dependence.

1 What is being shown

The KFE and closure conditions force both coefficients to be affine (linear wrt the IRFR) in the state variable x.

2 The key equation / identity

KFE convention used here:

$$\varphi_t = (g^2\varphi)_{xx} + (f\,\varphi)_x$$

Affine (linear wrt the IRFR) conclusion:

$$\boldsymbol{g(x,t,T) = a(\tau)x + b(\tau)}$$
$$\boldsymbol{f(x,t,T) = a_1(\tau)x + b_1(\tau)}$$
$$\boldsymbol{\tau = T - t}$$

Notation convention used here versus SDE convention:

$$f(x,t,T) = -\mu,$$
$$g^2(x,t,T) = \frac{\sigma^2}{2}$$

3 The intuition and payoff

The model does not guess an affine rate process. The affine state dependence is the closed form produced by rolling invariance plus KFE closure.

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Affine closure without the full proof

The deck can show the mathematical process, while leaving the algebra in the paper.

Process

1. **Take the KFE in x**

   Start from the density equation for the IRFR state variable x.

2. **Change variable to y**

   Use the chain rule so the same law is written in y-coordinates.

3. **Match KFE terms**

   Compare coefficients of 2nd, 1st and 0th order derivatives of φ.

4. **Separate x and y**

   The matched equations split into maturity/state components.

5. **Read off closure**

   The simplest closed admissible branch is affine in the rate state.

KFE-compatible affine branch

With remaining maturity written as T - t:

$$f(t,T,x) = a_1 x + b_1 (T - t)$$

$$g(t,T,x) = a\,x + b(T - t)$$

The state dependence is linear; the remaining-maturity dependence is carried by the intercept loadings.

Message: affine form is not guessed; it is the simplest KFE-compatible closure across rolling maturities.

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Coefficient matching

The proof becomes a short roadmap: match terms, separate variables, then retain the closed affine branch.

1	**Second-order term**	Match the coefficient of $\varphi_{xx}/\ \varphi_{yy}$ after the chain-rule change.	Diffusion loading behaves like a state-space scaling object.
2	**First-order term**	Use the transformed drift and derivative terms after the diffusion scaling is fixed.	The drift and diffusion loadings must separate into state and maturity pieces.
3	**Zeroth-order term**	Match the remaining density term in the transformed KFE.	The same separation closes the affine branch for f and g.

Resulting admissible structure

$$f(x,t,T) = a_1 x + b_1(T-t)$$
$$g(x,t,T) = ax + b(T-t)$$

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Use this as the proof-section substitute; detailed algebra can remain in the paper or appendix.

From affine state dependence to exponential maturity loadings

The affine proof fixes x-dependence; the intercept terms still need maturity structure.

1 What is being shown

The transformed KFE has already delivered the closed affine branch:
So x-dependence is fixed; the remaining task is to determine the maturity intercepts.

2 Key equations

Start from the affine forms:

Shown so far

$$g(x,t,T) = ax + b(t,T)$$
$$f(x,t,T) = a_1 x + b_1(t,T)$$

Rolling-maturity invariance means the intercepts depend on time only through remaining maturity, τ .

3 Intuition and payoff

The state variable x gives the local rate level. The intercepts $b_1(\tau)$ and $b(\tau)$ encode deterministic maturity geometry. The next step is to fix them using equilibrium transport.

$$\tau = T - t, \qquad b(t,T) = b(\tau), \qquad b_1(t,T) = b_1(\tau)$$

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Message: affine in state first; exponential maturity loadings second.

Drift intercept: equilibrium transport fixes $b_1(\tau)$

The first moment tells us what the drift intercept must be once the equilibrium curve is chosen.

1 What is being shown

At equilibrium, the expected IRFR is the maturity-indexed curve $x_e(\tau)$. The drift intercept must make the first-moment equation consistent with that curve.

2 Key identity: Axiom 3 applied to the first moment

First-moment equation:

$\partial_t E[x_{t,T}] = -a_1 E[x_{t,T}] - b_1(\tau)$

At equilibrium:

$E[x_{t,T}] = x_e(\tau)$,

Using Axiom 3:

$\partial_t x_e(\tau) = -x_e'(\tau)$

3 Intuition and payoff

The intercept is not arbitrary: it is the deterministic term required so that the affine drift rolls along the equilibrium mean curve.

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$$-x_e'(\tau) = -a_1 x_e(\tau) - b_1(\tau) \quad \Rightarrow \quad b_1(\tau) = x_e'(\tau) - a_1 x_e(\tau)$$

$$If \quad \boldsymbol{x_e(\tau) = x_\infty + Ae^{-\lambda^2\tau}}, \quad then \quad b_1(\tau) = -a_1 x_\infty - (a_1 + \lambda^2)Ae^{-\lambda^2\tau}$$

$$f(x,t,T) = a_1 x - a_1 x_\infty - (a_1 + \lambda^2)Ae^{-\lambda^2\tau}$$

Diffusion intercept: the shifted loading fixes $b(\tau)$

The diffusion intercept is justified by applying the same Axiom 3 transport logic to variance, not by importing a new assumption.

1 What is being shown

Write the diffusion loading around a deterministic maturity component. This makes the stochastic loading proportional to the shifted state.

2 Key identity: Axiom 3 applied to the second moment / variance

Affine diffusion loading:

$g(x,t,T) = ax + b(\tau)$

Shifted-state form:

$g(x,t,T) = ax - ax_\infty - (a + \lambda^2/a)Aexp[-\lambda^2\tau]$

Therefore

$b(\tau) = -ax_\infty - a(1 + \lambda^2/a^2)Aexp[-\lambda^2\tau]$

3 Intuition and payoff

The same maturity geometry fixes the diffusion intercept. Writing the noise in shifted-state form makes $b(\tau)$ inherit the same deterministic maturity structure as the drift intercept.

$$g(x,t,T) = ax - ax_\infty - (a + \lambda^2/a)Aexp[-\lambda^2\tau]$$

Working-paper intuition
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Message: affine state dependence comes first; equilibrium and rolling maturity give the exponential intercept structure.

Amplitude A: positivity fixes the exponential equilibrium branch – Apply Axiom 4 to drift

The equilibrium curve is exponential in remaining maturity, but the amplitude is not free once the short boundary is imposed.

1 **Start with the equilibrium curve**

$$\lambda^2 = ra^2$$
$$\tau = T - t$$
$$x_e(\tau) = x_\infty + A exp[-\lambda^2\tau]$$

2 **Use the interim diffusion loading**

$$\mathrm{f}(x, t, T) = a_1 x - a_1 x_\infty - (a_1 + \lambda^2/a_1) A exp[-\lambda^2\tau]$$

3 **Impose positivity at zero maturity**

At
$T = t,$
$x_{t,t} = 0$
$exp[-\lambda^2\tau] = 1.$

Set $\mathrm{f}(0, t, t) = 0$. Application of Axiom 4

$$0 = -a_1 x_\infty - a_1\left(1 + \lambda^2/a_1\right) A$$
$$A = -x_\infty/\left(1 + \lambda^2/a_1\right)$$

$$\Rightarrow f(x, t, T) = a_1 x - a_1 x_\infty + a_1 x_\infty e^{-\lambda^2\tau}$$

Working-paper intuition
not a peer-reviewed claim

Payoff: the short-end positivity condition fixes the amplitude of the exponential maturity component.

Amplitude A: positivity fixes the exponential equilibrium branch – Apply Axiom 4 to diffusion

The equilibrium curve is exponential in remaining maturity, but the amplitude is not free once the short boundary is imposed.

1 **Start with the equilibrium curve**

$$\lambda^2 = ra^2$$
$$\tau = T - t$$
$$x_e(\tau) = x_\infty + A exp[-\lambda^2\tau]$$

2 **Use the interim diffusion loading**

$$g(x,t,T) = ax - ax_\infty - (a + \lambda^2/a)A exp[-\lambda^2\tau]$$

3 **Impose positivity at zero maturity**

At
$T = t,$
$x_{t,t} = 0$
$exp[-\lambda^2\tau] = 1.$

Set $g(0,t,t) = 0$. Application of Axiom 4.

$$0 = -ax_\infty - a(1 + \lambda^2/a)A$$
$$A = -x_\infty/(1 + \lambda^2/a)$$

At zero maturity, admissibility requires not only vanishing diffusion loading but also no no deterministic drift away from the boundary

$$\Rightarrow g(t,T,x) = ax - ax_\infty + ax_\infty e^{-\lambda^2\tau}$$

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Payoff: the short-end positivity condition fixes the amplitude of the exponential maturity component.

Axiom 4 fixes the same amplitude twice

Boundary consistency gives a more direct route to the baseline identification.

1. **Start with the same equilibrium curve**

Before applying the short-end boundary, the equilibrium curve has one shared amplitude A.

$$x_e(\tau) = x_\infty + A\exp[-\lambda^2\tau]$$
$$\tau = T - t, \qquad x_e(0) = x_\infty + A$$

2. **Apply Axiom 4 to the diffusion loading**

The boundary condition g(0,t,t)=0 fixes A through the diffusion scale.

$$A = -\left[\frac{a^2}{(a^2 + \lambda^2)}\right] x_\infty$$

3. **Apply Axiom 4 to the drift loading**

The boundary condition f(0,t,t)=0 fixes the same A through the drift scale.

$$A = -\left[\frac{a_1}{(a_1 + \lambda^2)}\right] x_\infty$$

Core intuition

The same equilibrium curve cannot carry two different short-boundary amplitudes.

If Axiom 4 is applied consistently to both drift and diffusion, the two expressions for A must agree.

This makes the identification $a_1 = a^2$ feel like boundary consistency, not an arbitrary simplification.

Payoff: Axiom 4 does more than impose positivity; it ties the drift and diffusion scales together.

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Boundary consistency gives $a_1 = a^2$

The two Axiom 4 boundary calculations must describe the same equilibrium curve.

Equate the two values of A

$$a^2 / (a^2 + \lambda^2) = a_1 / (a_1 + \lambda^2)$$

$$a^2 a_1 + a^2\lambda^2 = a_1 a^2 + a_1\lambda^2$$

$$a^2\lambda^2 = a_1\lambda^2$$

assuming $\lambda^2 \neq 0$,

$$a_1 = a^2$$

What is being shown

Axiom 4 can be applied separately to the diffusion boundary and the drift boundary. Each fixes the same amplitude A.

Key interpretation

Consistency requires the drift-side amplitude and diffusion-side amplitude to agree. This forces the drift scale and diffusion scale to match.

Presentation message

The baseline $a_1 = a^2$ is now a boundary-consistency result. The variance-clock argument can remain as supporting intuition, but it no longer needs to carry the proof burden.

Baseline identification: $a_1 = a^2$

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Full specification of parameters

All dynamics have been distilled into remaining parameters

There are two rate parameters $a^2 and\ \lambda^2$ as well as x_∞

All key properties can be expressed in terms of the parameters and some representation of initial condition

Functional form of key properties

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Application of Baseline Parameters to Key Relationships

The moment formulas are consequences of the probability law.

1 What is being shown

Having established that

$$f(x,t,T) = a^2x - a^2x_\infty + a^2x_\infty e^{-\lambda^2\tau}$$

$$g(t,T,x) = \ ax \ - ax_\infty + ax_\infty e^{-\lambda^2\tau}$$

And applying the general r-branch

We are able to provide all the key relationships shown opposite

These relationships should adhere to all the properties required by the Axioms

Therefore, the relationships should form a well-grounded set of results all of which can be related back to the axiomatic roots

As a note, these results have been shown to be consistent with the Axioms, not necessarily unique to them.

2 The key equation / identity

Expectation:

$$E\left[x_{t,T}\right] = x_\infty - \frac{x_\infty}{(1+r)} e^{[-ra^2(T-t)]} + C_r e^{[-a^2(T+t-2t_0)]}$$

Expectation Rate:

$$\partial_t E\left[x_{t,T}\right] = -[ra^2x_\infty/(1+r)]exp[-ra^2(T-t)] - a^2C_r exp[-a^2(T+t-2t_0)]$$

Variance-rate:

$$\partial_t V = C_r\left[\exp[-a^2(T+t-2t_0)] + r\tfrac{x_\infty}{1+r}\exp[-r\,a^2(T-t)]\right]^2$$

Equilibrium Curve Shape

$$x_e(T-t) = x_\infty\left[1 - \left[\frac{1}{(1+r)}\right] exp[-ra^2(T-t)]\right]$$

IRFR Curve Shape

$$x_{t_0,T} = x_\infty - [x_\infty/(1+r)]exp[-ra^2(T-t_0)] + C_r exp[-a^2(T-t_0)]$$

where:

$$C_r \coloneqq \ x_{t_0,t_0} - r\tfrac{x_\infty}{1+r}$$

$$r \coloneqq \frac{\lambda^2}{a^2}$$

Working-paper intuition
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Equilibrium curve and the initial forward curve

Once A is fixed, the general r-baseline produces two deterministic maturity loadings.

1 Equilibrium curve

$$x_{e,0} = \left[\frac{r}{(1+r)}\right] x_\infty$$
$$x_e(T-t) = x_\infty \left[1 - \left[\frac{1}{(1+r)}\right] exp[-ra^2(T-t)]\right]$$

2 Initial curve at t_0

$$x_{t_0,T} = x_\infty - [x_\infty/(1+r)]exp[-ra^2(T-t_0)] + \left(x_{t_0,t_0} - [r/(1+r)]x_\infty\right) exp[-a^2(T-t_0)]$$

3 Compact notation

$$C_r := x_{t_0,t_0} - [r/(1+r)]x_\infty$$
$$\tau_0 := T - t_0 \ , r := \frac{\lambda^2}{a^2}$$

Interpretation

The long end is set by x_∞. The short-end is set by Axiom 4.The ratio r controls the second maturity clock.

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Message: one stochastic factor, but two deterministic maturity loadings when r is not one.

Expected value rewritten in the general r-baseline form

Substitute the initial curve into the expectation solution after imposing $a_1 = a^2$.

1 **Start from the expectation solution**

$$E[x_{t,T}] = x_e(T-t) + \left(x_{t_0,T} - x_e(T-t_0)\right) exp[-a^2(t-t_0)]$$

2 **Substitute the initial curve**

$$E[x_{t,T}] = x_\infty - [x_\infty/(1+r)] exp[-ra^2(T-t)] + C_r exp[-a^2(T+t-2t_0)]$$

3 **Derivative with respect to t**

$$\partial_t E[x_{t,T}] = -[ra^2 x_\infty/(1+r)] exp[-ra^2(T-t)] - a^2 C_r exp[-a^2(T+t-2t_0)]$$

Payoff

Expectation now has the two maturity clocks that later generate the admissible curve shapes. Calendar-time convergence and maturity structure remain separated.

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Message: this is the expected IRFR after baseline drift-diffusion identification, with r retained.

Instantaneous variance-rate in the general r-baseline form

The baseline removes the homogeneous V term; the remaining variance-rate is the square of the two-loading structure.

1 General variance relation

$$\partial_t V = 2\left(aE\left[x_{t,T}\right] + b\right)^2$$

2 General r-baseline variance-rate

$$\partial_t Var\left(x_{t,T-t}\right) = 2a^2[A_r exp[-a^2(T-t)] + B_r exp[-ra^2(T-t)]]^2$$

3 Definitions and local volatility

$$A_r \coloneqq C_r exp[-2a^2(t-t_0)]$$
$$B_r \coloneqq [r/(1+r)]x_\infty$$
$$\sigma_{loc}(\tau) = \sqrt{(\partial_t Var)}$$

Interpretation

This is instantaneous local variance-rate, not the stationary equilibrium volatility curve. When the two terms have the right signs and scales, the local volatility can be humped.

Working-paper intuition
not a peer-reviewed claim

Message: the variance formula inherits the same two-loading maturity geometry as the expectation.

A special note on: Market Signal-to-Noise Ratio (MSNR)

The moment formulas set up the signal-to-noise result.

1

What has been shown

Expectation gives the signal branch; variance-rate gives the instantaneous noise scale.

2

What comes next

MSNR forms the ratio of expected instantaneous return to the square-root variance-rate.

3

Why it matters MSNR is globally constant

$$\mathrm{MSNR} = \frac{a}{\sqrt{2}}$$

Bridge identity

Signal scale $= \partial_t E[x_{t,T}]$ — *The instantaneous expected value – related to the P&L of the IRFR*

Noise scale $= \sqrt{\partial_t \operatorname{Var}(x_{t,T})}$ — √[*The instantaneous variance value* | – *related to the √variance of that P&L*

$\mathrm{MSNR} = \dfrac{\text{Signal}}{\text{Noise}} = \dfrac{a}{\sqrt{2}}$ — *Heuristically, a ratio of the local return/volatility*

No dependence on $x,\ t,$ or T.

Presentation reminder

Since the IRFR is not itself a traded asset, we do not call this a Sharpe ratio. Instead, we examine the related Market Signal-to-Noise Ratio. Critically, this quantity is independent of:

- state x,
- calendar time t,
- or maturity T.

It is tied to a, not to a maturity-by-maturity curve.

Working-paper intuition
not a peer-reviewed claim

Payoff: Section 4 supplies the objects needed for Section 5, the MSNR result.

Functional form of key properties presented

From these functional forms we can…

Re examine and present graphically…

IRFR admissible curves and volatility

Stylised graphical representations

Working-paper intuition
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Model outputs: admissible IRFR curve shapes

Having built the framework, we can now show what the general r-branch allows.

The curves on the right illustrate:
• the equilibrium benchmark;
• an admissible curve with an interior maximum;
• an admissible curve with an interior minimum.

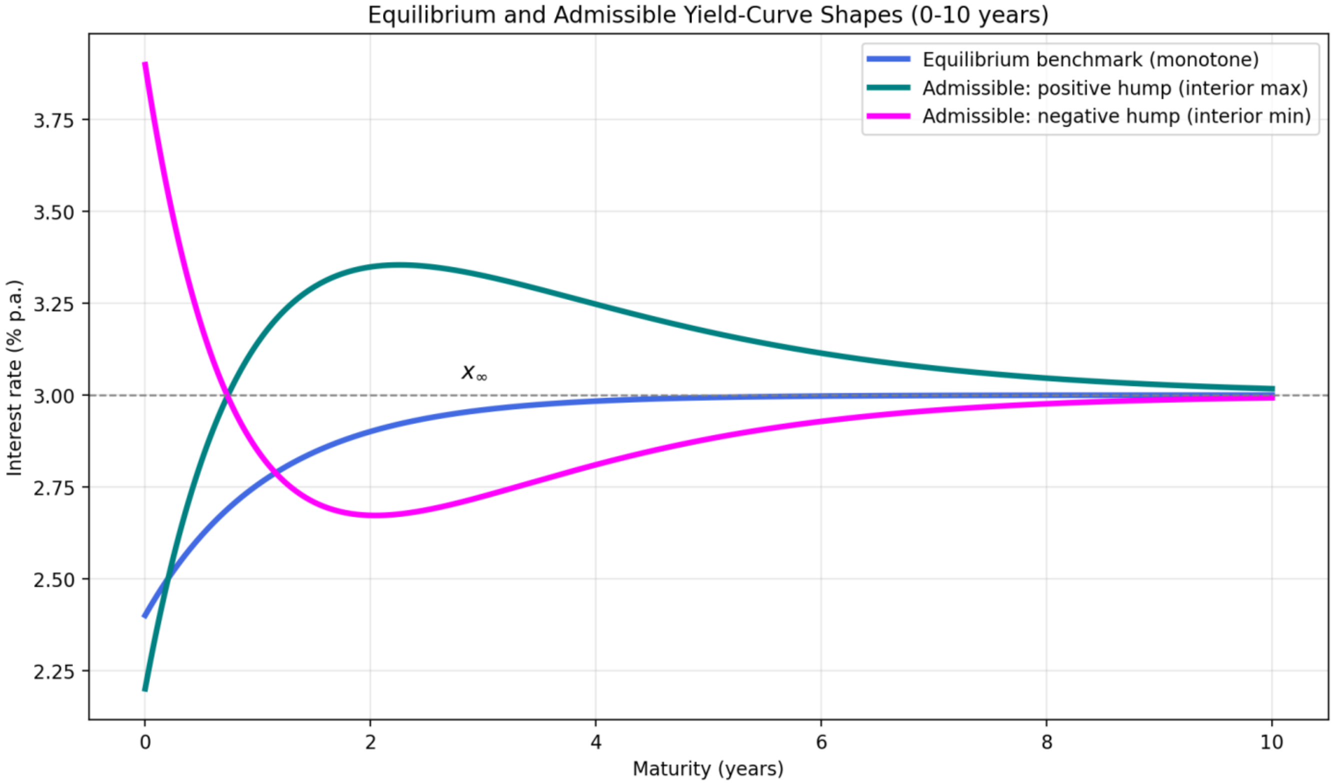
The mechanism is the interaction of two deterministic maturity loadings:

$$\exp[-a^2\tau] \quad and \quad \exp[-r\, a^2\tau]\,.$$

No second Brownian factor is added. The randomness remains one-dimensional; the extra shape comes from deterministic maturity geometry.

Important: these are stylised admissible shapes, not a claim that all observed humps or dips arise without policy, liquidity or market-segmentation effects.

Payoff: one factor in randomness does not imply one loading in maturity.



Working-paper intuition not a peer-reviewed claim

Humped instantaneous local volatility

The same two-loading structure can generate a humped maturity profile for instantaneous local volatility.

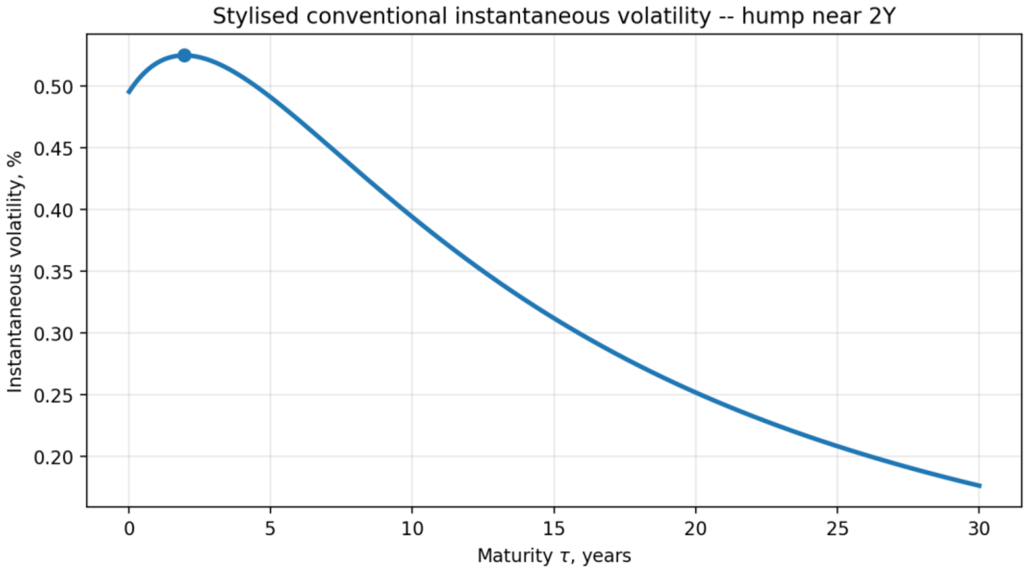
The plotted object is:
$\sigma_{loc}(\tau) = \sqrt{[\, \partial t \; Var(x_{t,T})]}.$

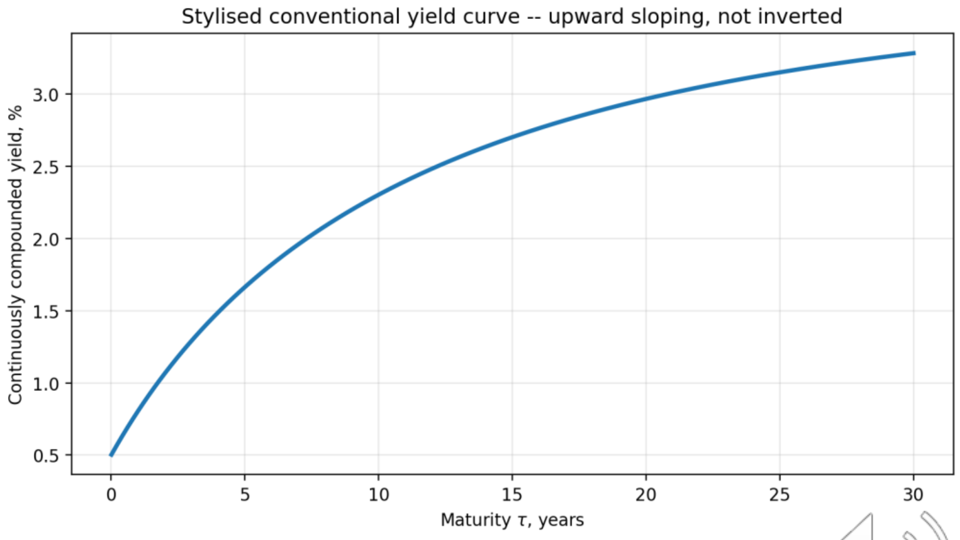
This is not the stationary equilibrium volatility curve. At stationary equilibrium, the volatility profile is monotone. The hump shown here belongs to an admissible non-stationary rolling-maturity branch.

The associated IRFR curve is shown below. It has been chosen without an interior maximum to make the point clear: a humped volatility profile does not require a humped IRFR curve.

Payoff: the hump comes from the deterministic maturity profile inside the variance-rate, not from a second stochastic factor.

Working-paper intuition
not a peer-reviewed claim





Solve for the PDF

The KFE selects a shifted inverse-gamma equilibrium density.

1 What is being shown

Move into the shifted coordinate, use the reciprocal variable, and the Frobenius recurrence selects the inverse-gamma branch.

2 The key equation / identity

Shifted coordinate:

$$\widetilde{x_{t,T}^{(r)}} = x_{t,T} - x_\infty(1 - \exp[-ra^2(T-t)])$$

Distribution:

$$\widetilde{x_{t,T}^{(r)}} \sim \mathrm{InvGamma}(r+2,, r x_\infty \exp[-ra^2(T-t)])$$

Density:

$$\varphi(\tilde{x}) = \frac{\beta^{r+2}}{\Gamma(r+2)} x^{-\widetilde{(r+3)}} \exp\left[-\frac{\beta}{\tilde{x}}\right]$$

$$\beta = r x_\infty \exp[-ra^2(T-t)]$$

3 The intuition and payoff

The density is solved, not assumed. The baseline order-three law is only the r = 1 branch of the broader family.

Working-paper intuition
not a peer-reviewed claim

Equilibrium density: the distributional payoff

The KFE gives more than moment equations: it selects a maturity-indexed equilibrium density.

Shift the state variable:

$$\tilde{x}^{(\mathrm{r})}_{\{t,T\}} = x_{\{t,T\}} - x_\infty(1 - \exp[-r\, a^2(T-t)]).$$

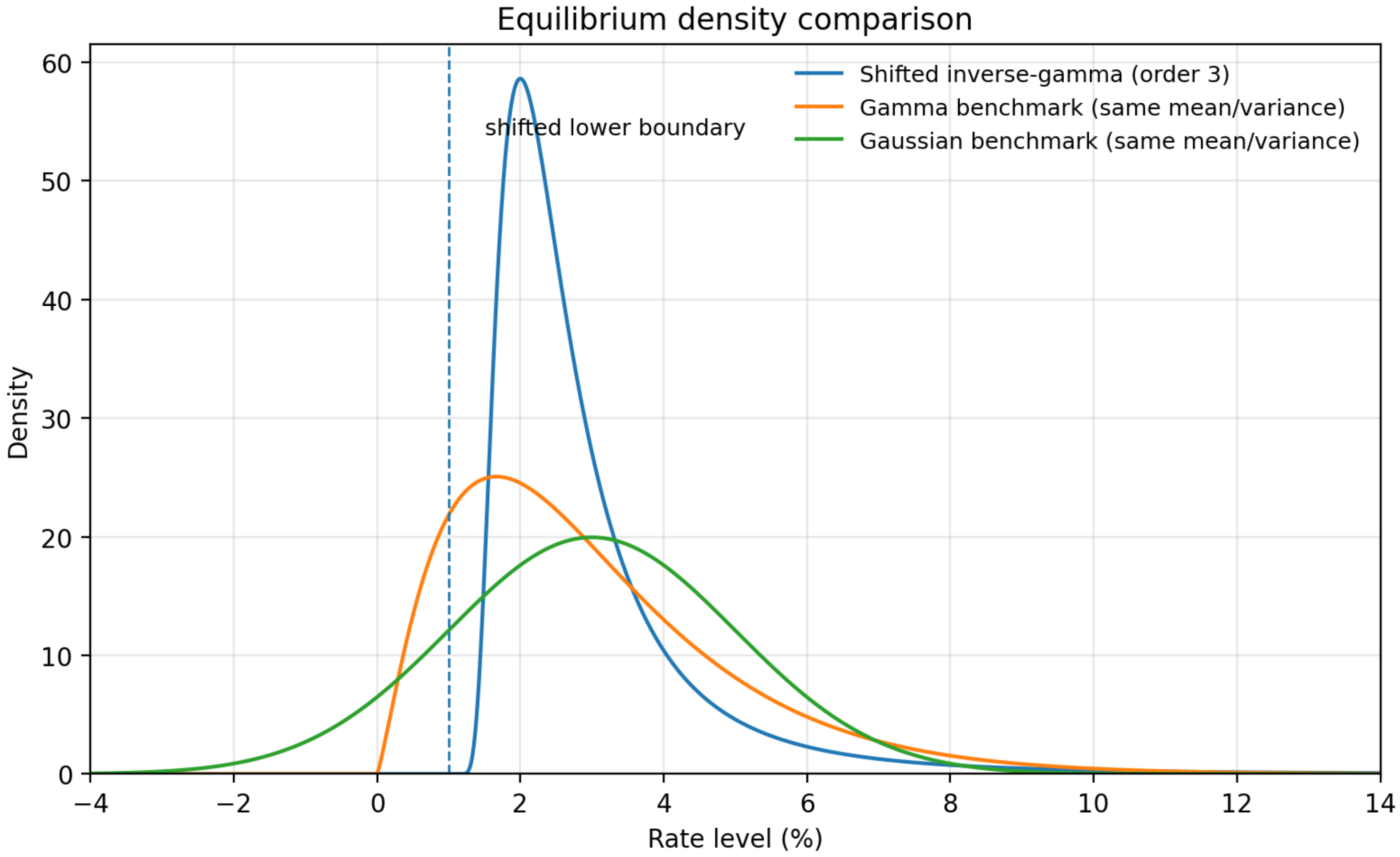
In this coordinate:

$$\tilde{x}^{(\mathrm{r})}_{\{t,T\}} \sim InvGamma(r+2, r\, x_\infty \exp[-r\, a^2(T-t)]).$$

When r=1, the shape parameter is 3: the baseline order-three shifted inverse-gamma branch.

Compared with Gaussian and gamma benchmarks, the shifted inverse-gamma keeps the positive boundary and gives a right-skewed, heavier-tailed rate distribution.

Payoff: the framework produces a full equilibrium law, not just a mean curve and variance formula.



Working-paper intuition
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Conclusion: what the IRFR framework gives us

A rolling primitive turns curve shape, local risk and equilibrium density into one distributional story.

1 Primitive

- The IRFR is chosen because its infinitesimal change has local P&L meaning.
- The object rolls naturally as calendar time advances and remaining maturity shortens.

2 Mechanics

- Axioms + KFE give affine state dependence and exponential maturity loadings.
- The general r-branch preserves one Brownian factor but allows two deterministic maturity clocks.

3 Outputs

- Expectations, variance-rate, admissible curve shapes and the equilibrium density come from the same framework.
- MSNR is global: tied to a, not to maturity or state.

Key takeaway: one factor in randomness does not imply one loading in maturity.

The framework stays one-factor, but the maturity geometry is rich enough to generate curve shape, local volatility shape and a shifted inverse-gamma equilibrium law.

Working-paper intuition
not a peer-reviewed claim

Thank you!

https://papers.ssrn.com/sol3/papers.cfm?abstract_id=6748347

https://sachinoza56-lgtm.github.io/IRFR-Paper/